Theory of parton distribution functions

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- Introduction: coefficient functions, partons distributions and α_s
- From low scale structure-function data to LHC predictions
- Higher-order parton evolution, with flavour and small-x effects
- Practical evolution: codes and benchmarks. Heavy quarks
- Recent parton fits, open issues and outlook: W-mass at the LHC

* Freely using results obtained with Sven Moch (DESY) and Jos Vermaseren (NIKHEF)

Parton distributions in collider physics

Search for Higgs Boson, new particles : highest possible energies

$\Rightarrow$ $p\bar{p}/pp$ **colliders: Tevatron (2 TeV), LHC (14 TeV)**

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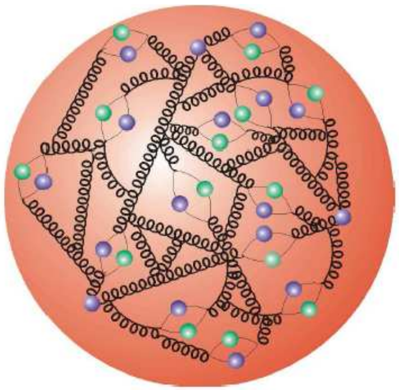
Proton: very complicated multi-particle bound state

"The good, the bad, and the baryon"

Colliders: wide-band beams of quarks and gluons



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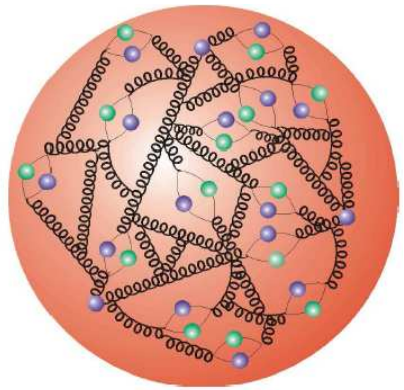
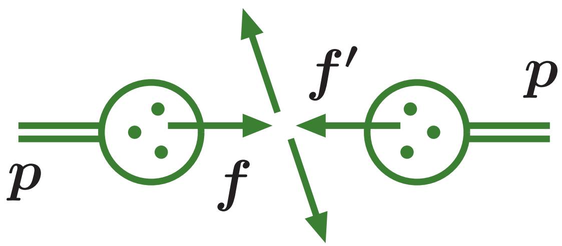


$$\sigma^{pp} = \sum f^p * f'^p * \hat{\sigma}^{ff'}$$



Hard interactions of protons:

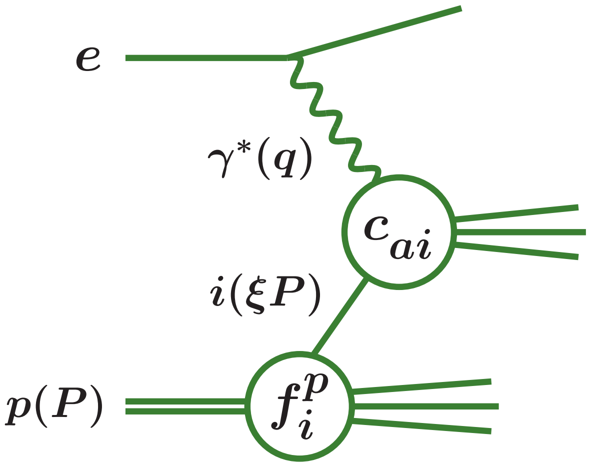
parton (q, g) distributions f^p

partonic cross sections $\hat{\sigma}^{ff'}$

$\Rightarrow$ Lepton-proton scattering: SLAC ep, CERN μp, νN, HERA, ...

Parton densities and hard processes in pQCD

Example: inclusive photon-exchange deep-inelastic scattering (DIS)



Hard scale, Bjorken variable x

$$Q^2 = -q^2$$

$$x = Q^2/(2P\cdot q)$$

$\mathcal{O}(\alpha_s^0)$: **quarks,** $x = \xi$ **(**$m = 0$**)**

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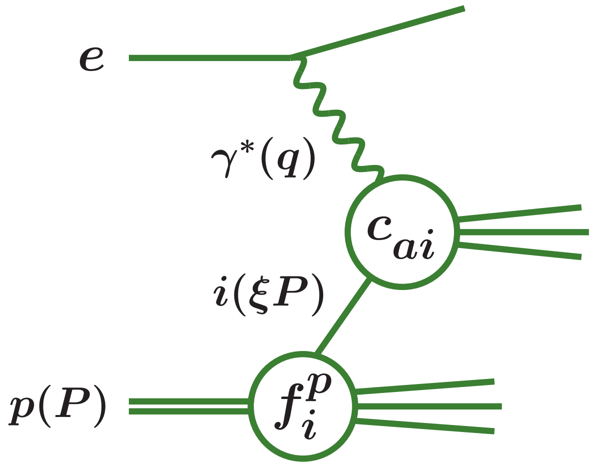

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Structure functions $F_{2,L}$ (at leading twist of operator-product exp.)

$$x^{-1}F_a^p(x,Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi}\, c_{a,i}\left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) f_i^p(\xi,\mu^2)$$

Coefficient functions: scheme, scale $\mu = \mathcal{O}(Q)$, Mellin convolutions

$1/Q^2$ corrections ('higher twists'): extract or suppress by data cuts

Parton densities and hard processes in pQCD

Parton distributions f_i: renormalization-group evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) = \sum_k \left[P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ **= Mellin convolution. Initial conditions incalculable in pert. QCD**

$\Rightarrow$ **predictions: fits of suitable reference processes, universality**

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Expansions in α_s: splitting functions P, coefficient functions c_a

$$P \;=\; \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \ldots$$

$$c_a \;=\; \underbrace{\alpha_s^{n_a}\left[\, c_a^{(0)}\right.}_{} + \alpha_s c_a^{(1)} + \alpha_s^2 c_a^{(2)} + \left.\ldots\,\right]$$

LO: approximate shape, rough estimate of rate

Parton densities and hard processes in pQCD

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Expansions in α_s: splitting functions P, coefficient functions c_a

$$\begin{aligned} P &= \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \ldots \\ c_a &= \underbrace{\alpha_s^{n_a} \Big[c_a^{(0)} + \alpha_s c_a^{(1)}}_{} + \alpha_s^2 c_a^{(2)} + \ldots \Big] \end{aligned}$$

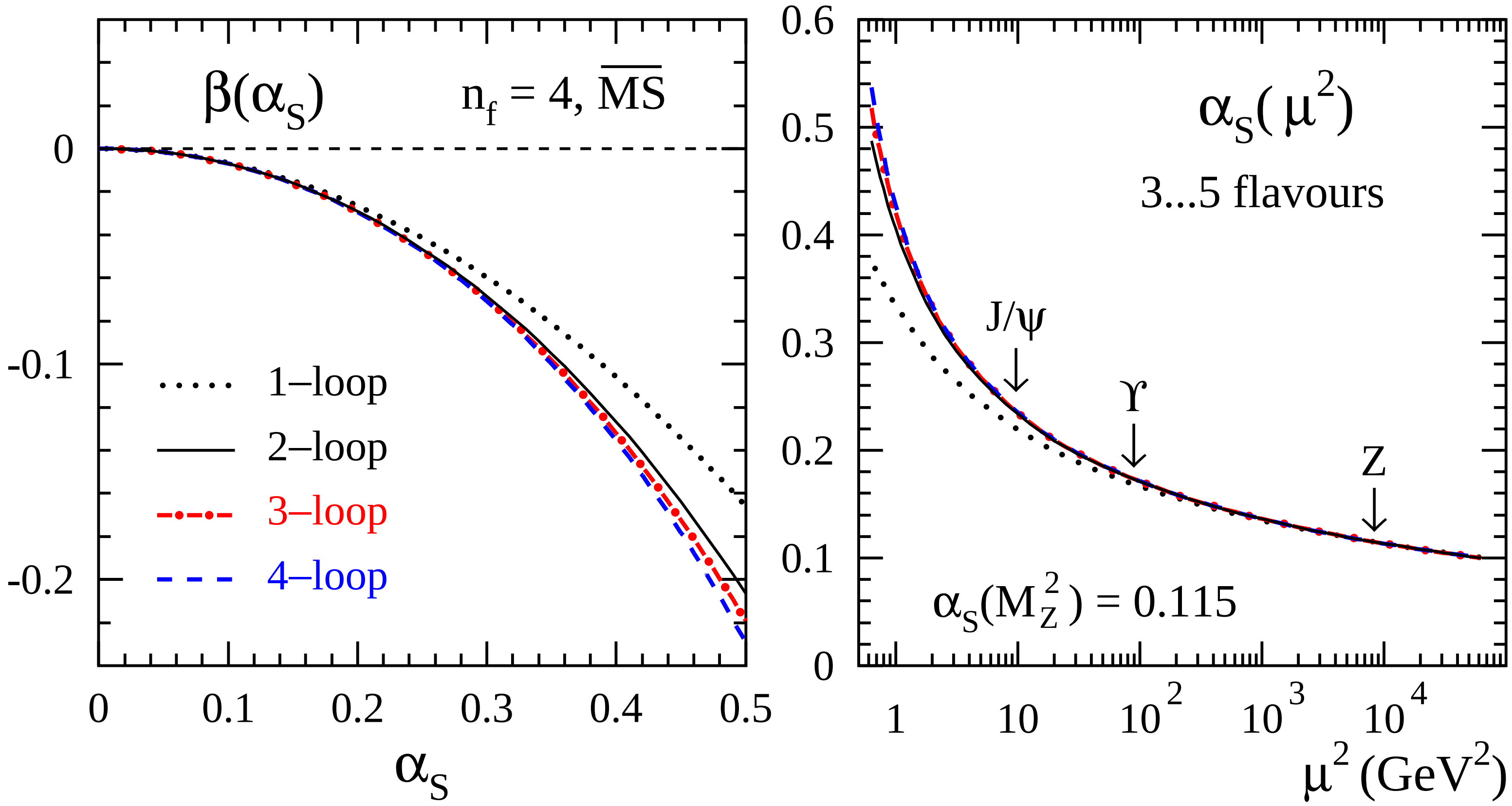
NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate $\Rightarrow$ precision physics

The running coupling in perturbative QCD

$$d\alpha_s / d\ln\mu^2 = -\beta_0\,\alpha_s^2 - \beta_1\,\alpha_s^3 - \beta_2\,\alpha_s^4 - \beta_3\,\alpha_s^5 - \ldots$$

N^{3}LO coefficient β_3 : van Ritbergen, Vermaseren, Larin (97); Czakon (04)

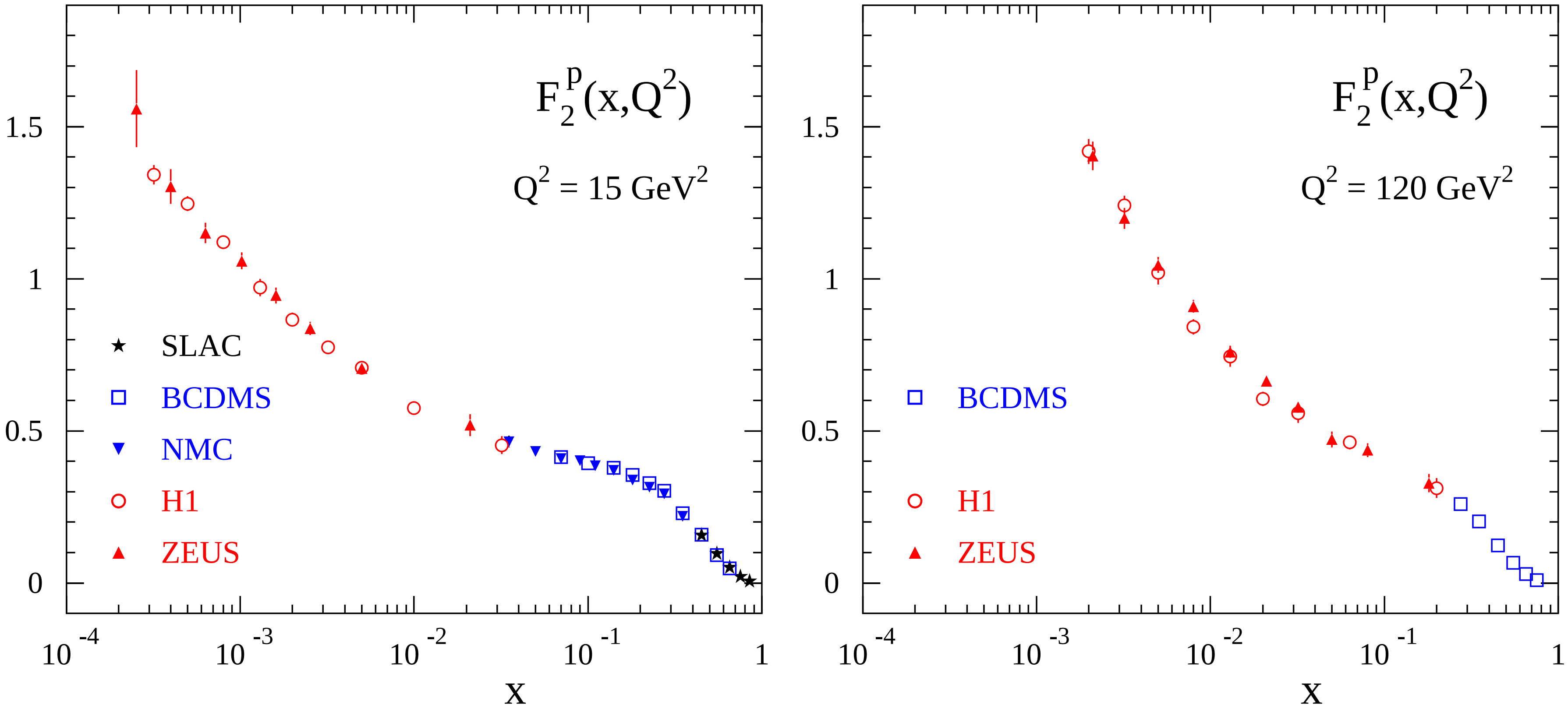


Perturbatively stable at $Q^2 > 1$ GeV^2. Boundary condition: exp. (+ lattice)

From structure functions to parton densities

Exp.: SLAC (e, 30 GeV), CERN (μ, 300 GeV), DESY ($e^{\pm}$, 30$\times$800 GeV)

Selected data on the proton structure function F_2



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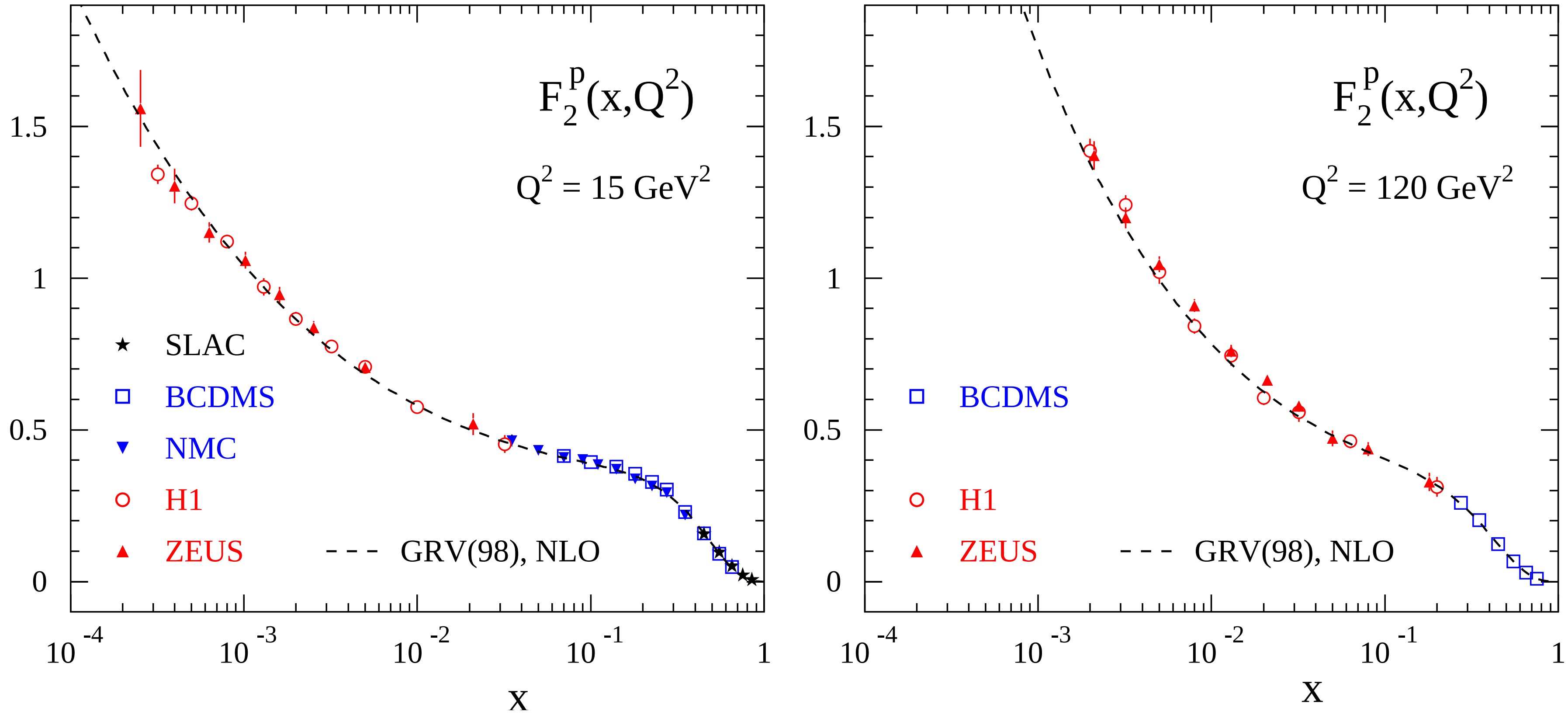
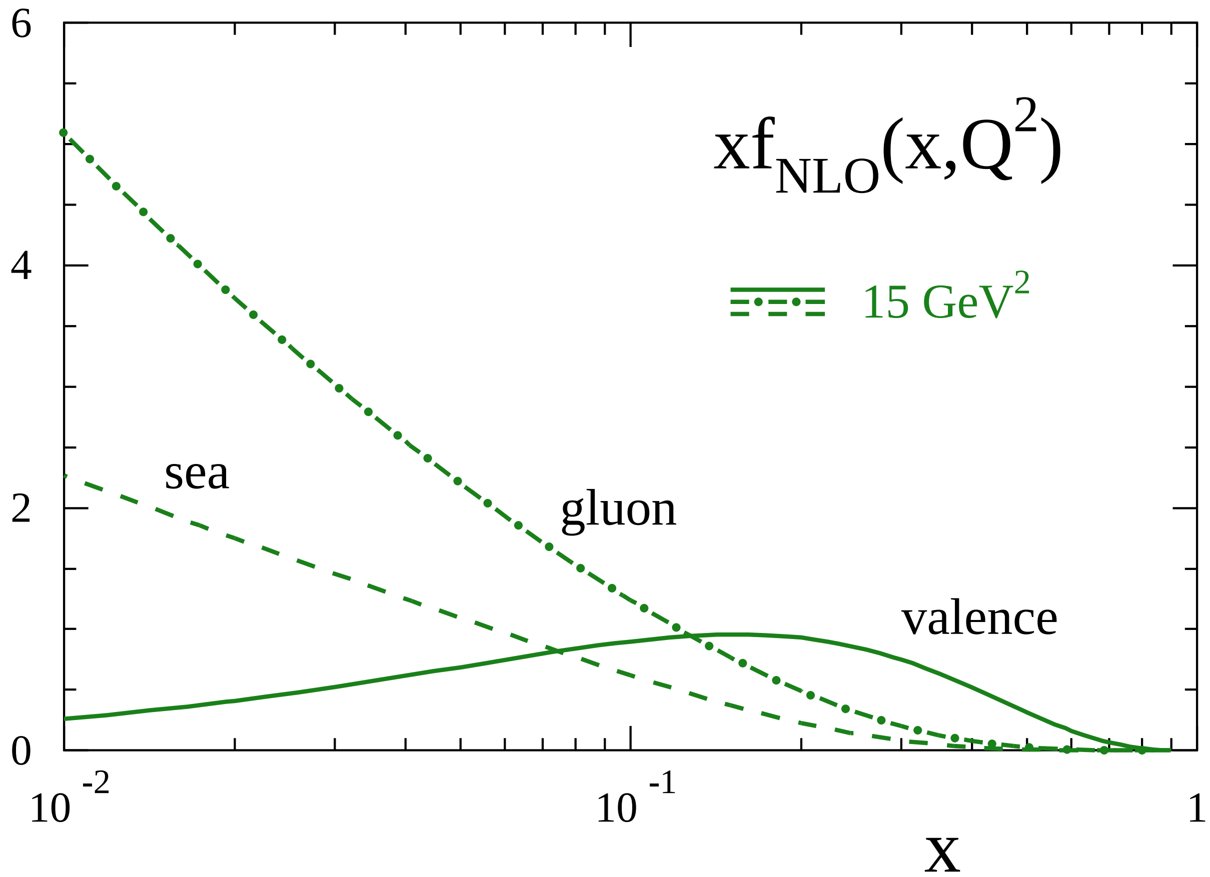


F_2 $\longrightarrow$ quark distributions,   scale dependence $\longrightarrow$ gluon distribution

The proton's parton densities, qualitatively

Valence: $q - \bar{q}$ $\leftrightarrow$ **additive quantum numbers. Quark sea:** $q = \bar{q}$ **parts**

Large x **(** $\overset{\text{now}}{\equiv} \xi$ **): valence** $\gg$ **glue** $\gg$ **sea. Small** x**: glue** $>$ **sea** $\gg$ **valence**



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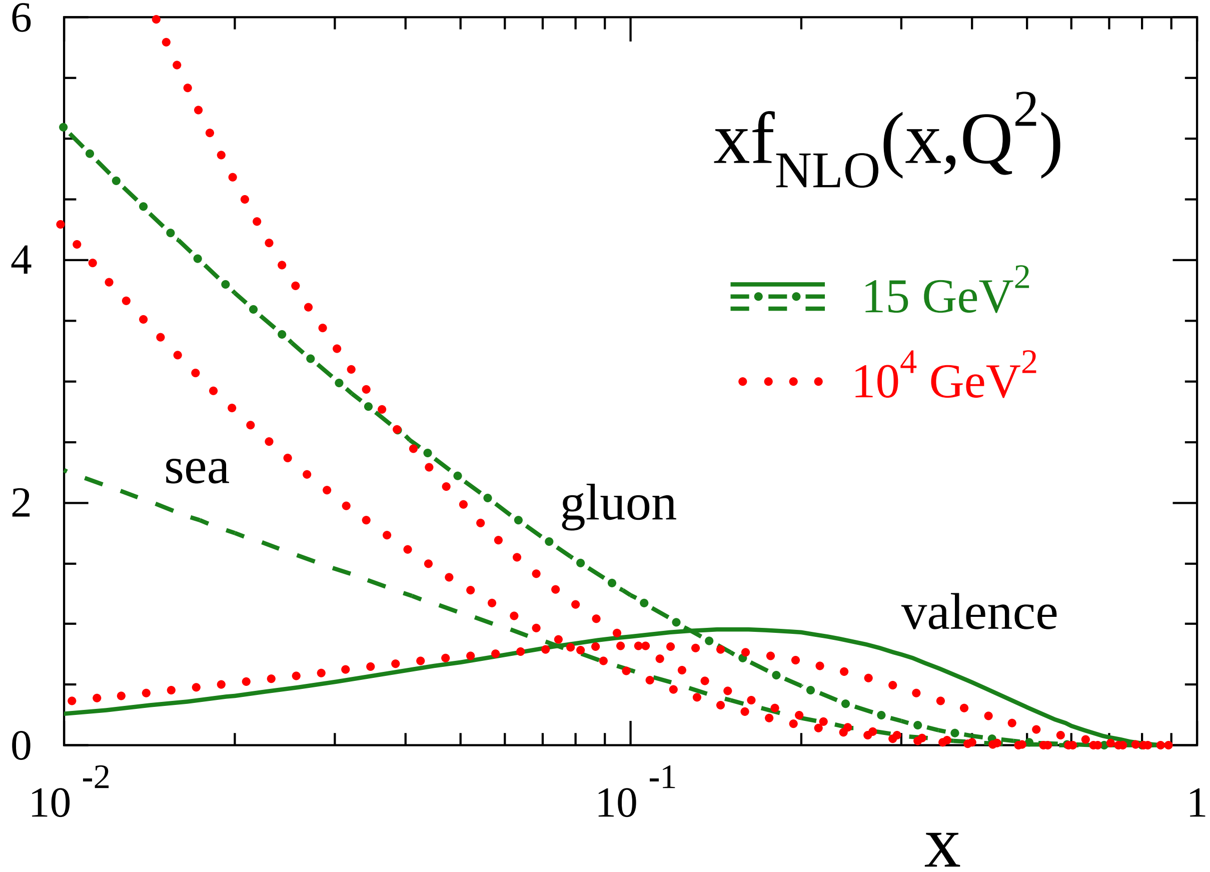
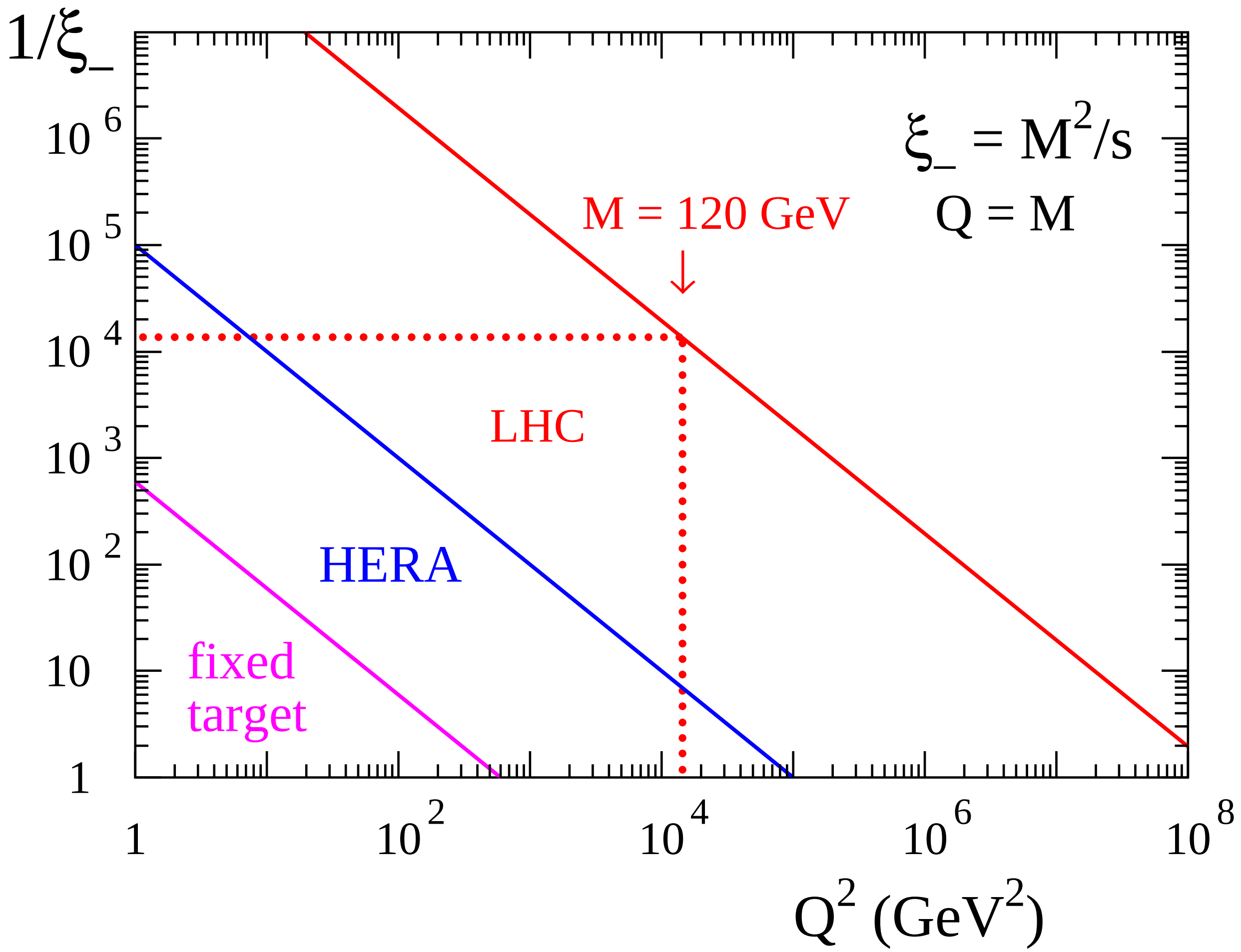


Scale dependence calculable : perturbative evolution equations

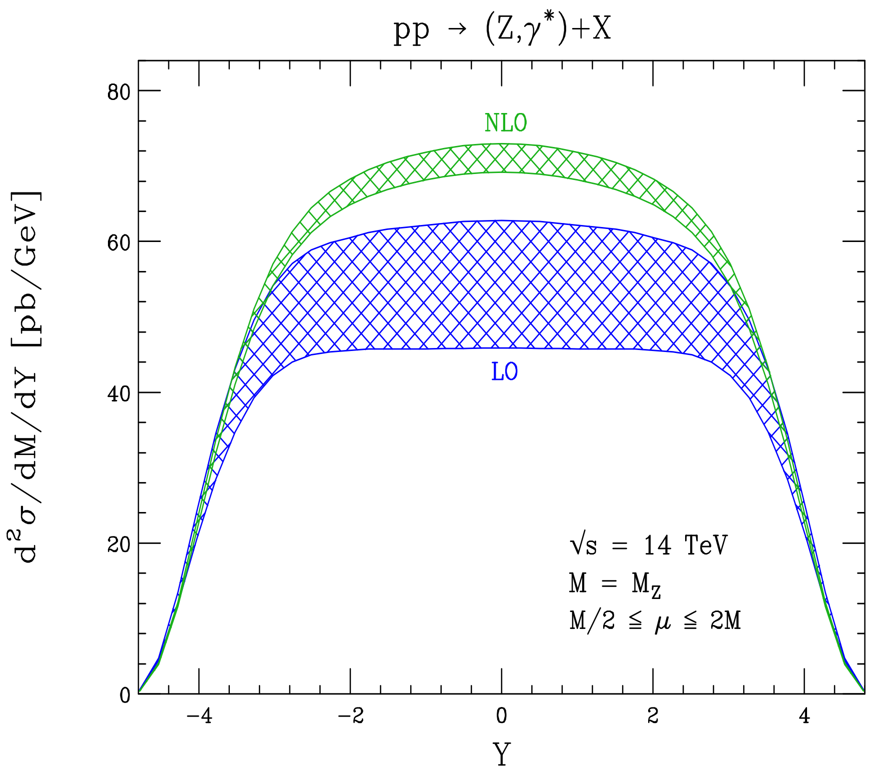
Parton evolution from HERA to LHC

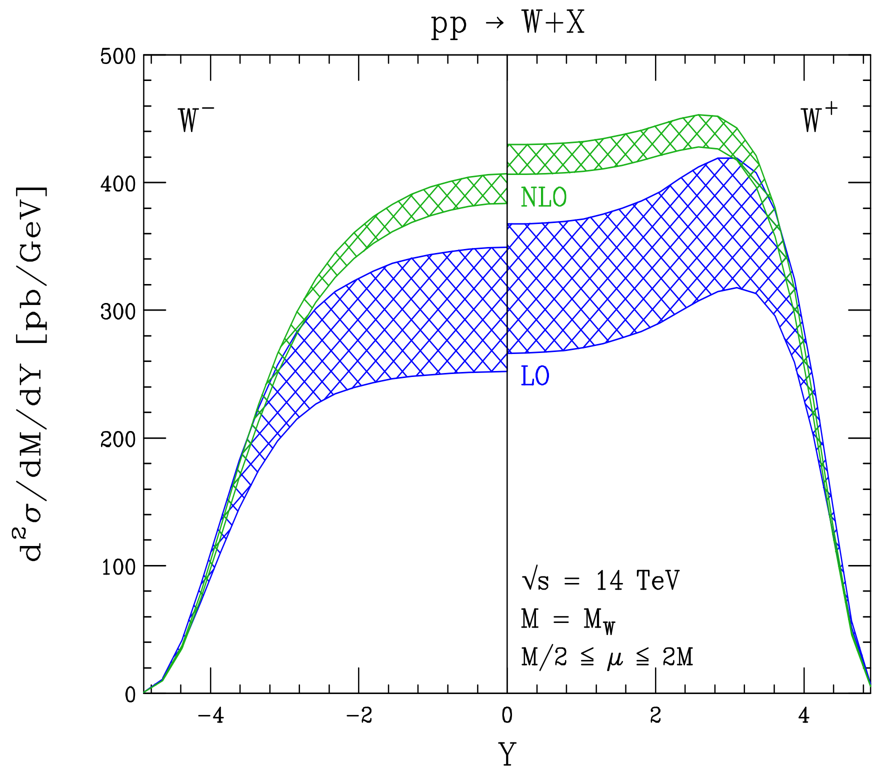
Kinematics: partons with momentum fractions $\xi_- < \xi < 1$ contribute



W/Z, H, top, new phys.: $\xi_- \gtrsim 10^{-4}$, can cut DIS at $Q^2 \approx 10$ GeV^2

Gauge boson production at the LHC





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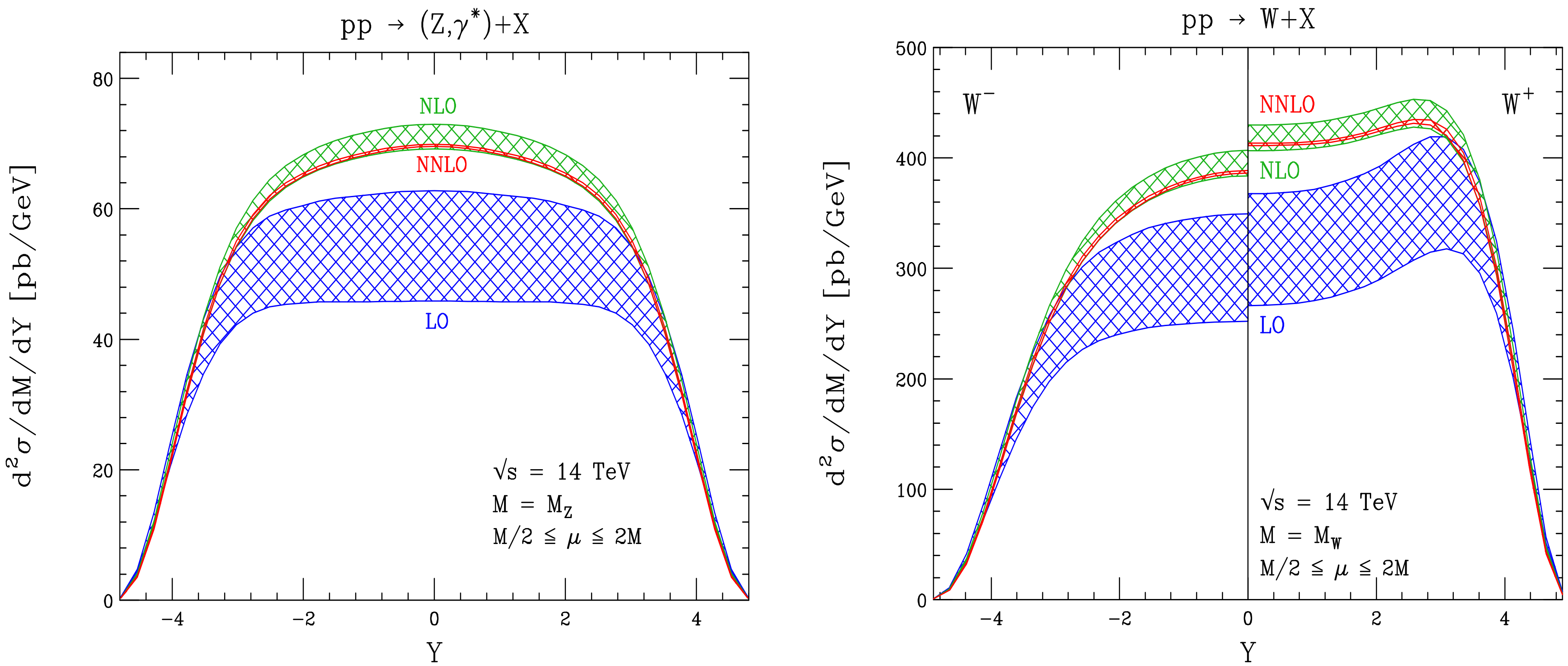


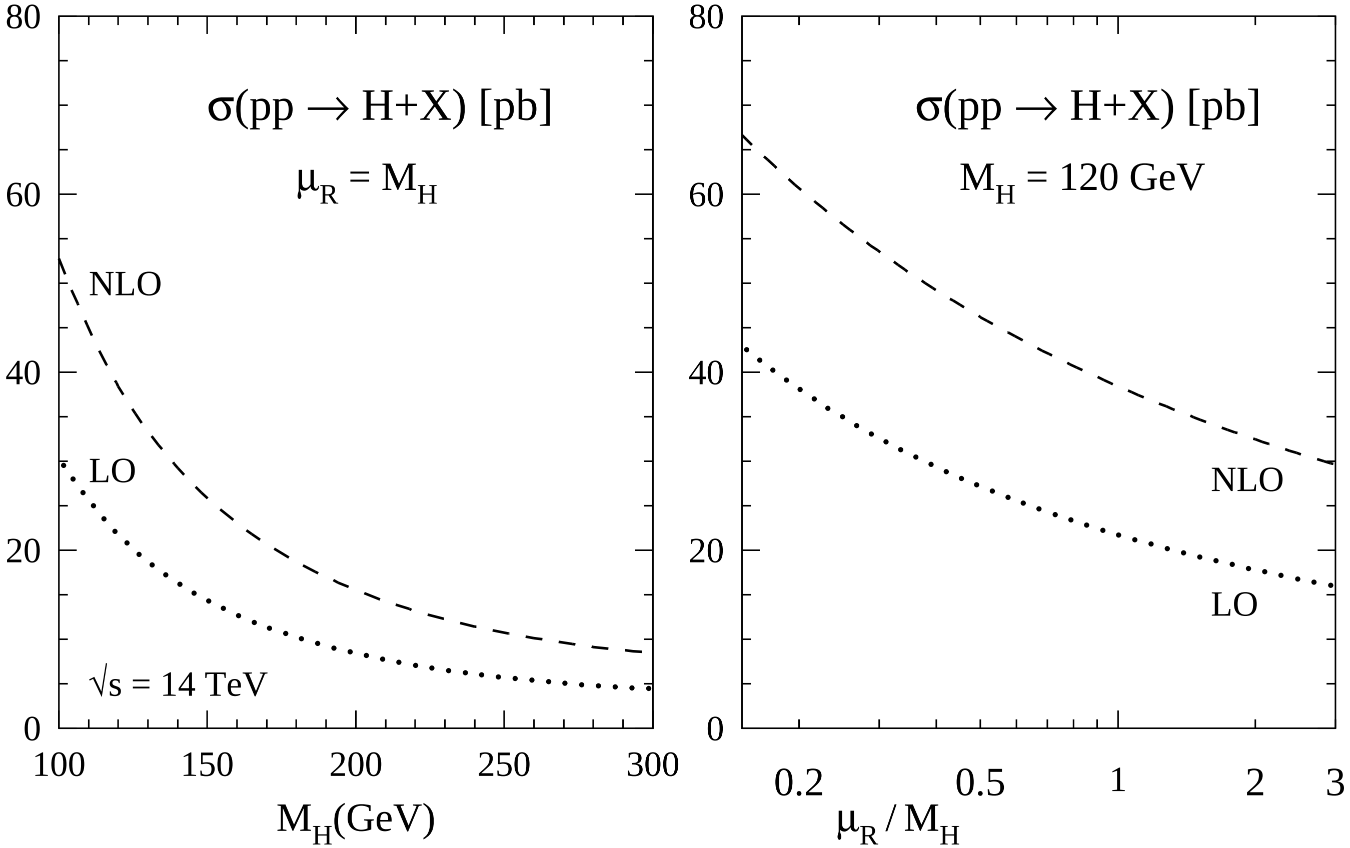
Rapidity-dependent $\hat{\sigma}_{NNLO}$: Anastasiou, Dixon, Melnikov, Petriello (03)

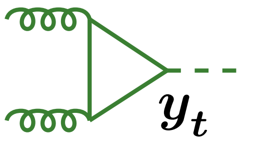
'Gold-plated' processes: NNLO perturbative accuracy better than 1%

$\Rightarrow$ check/improve high-scale parton densities (fixed in plots above)

Disclaimer (I): much lower accuracy possible

Example: total cross section for Higgs boson production at the LHC





Spira et al. (95)

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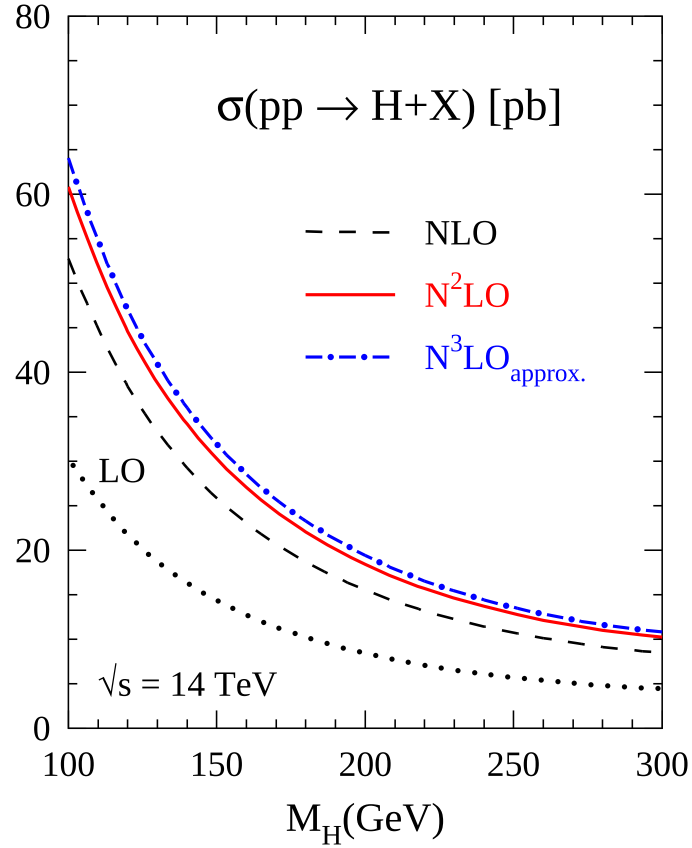
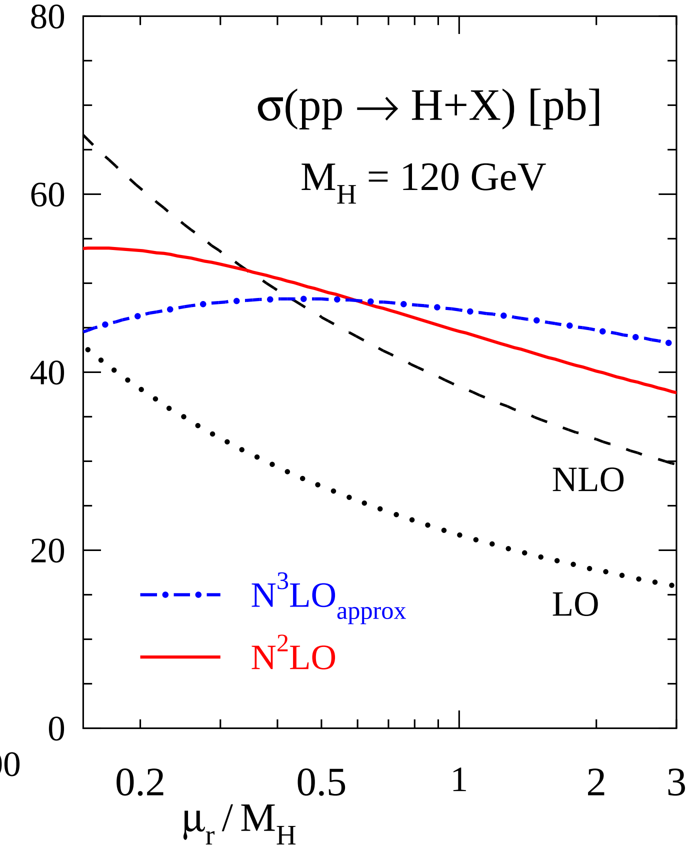
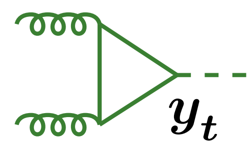






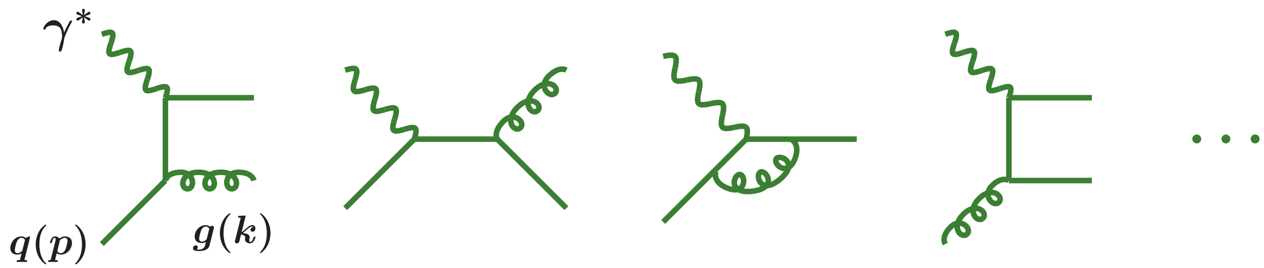
Moch, A.V. (05)

$\hat{\sigma}_{\mathrm{NNLO}}$ **: Harlander, Kilgore (02); Anastasiou, Melnikov (02, 05 [σ_{diff}])**

Higher-order uncertainties: $\sim 15\%$ at NNLO, $\sim 5\%$ at approx. N^3LO

Form mass singularities to splitting functions

First-order DIS:
(inclusive, $\int d^4k$)



Emissions collinear to the incoming partons: mass singularities

$$(p-k)^2 \;=\; -2|\vec{p}\,||\vec{k}|(1-\cos\vartheta) \;\xrightarrow{\vartheta\to 0}\; -|\vec{p}\,||\vec{k}|\,\vartheta^2 \;\;\Rightarrow\;\; \int d\vartheta/\ldots$$ **divergent**

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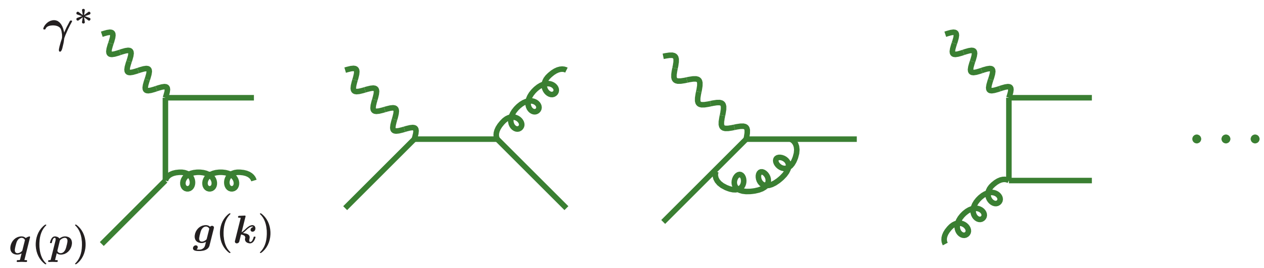

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Regularization (dim. $= 4 - 2\varepsilon$, poles $\sim 1/\epsilon$) and mass factorization

$$F_a(Q^2) \;=\; \hat{F}_{a,k}(\alpha_s(Q^2),\epsilon)\otimes \hat{f}_k \;=\; C_{a,i}(\alpha_s(Q^2)) \otimes \underbrace{\Gamma_{ik}(\alpha_s(Q^2),\epsilon)\otimes \hat{f}_k}_{f_i(Q^2)}$$

$C_{a,i}$ **: coefficient functions of** F_a

Γ_{ik} **:** $1/\epsilon$ **- poles (universal) + . . . ,** $\overline{\textbf{MS}}$ **scheme**

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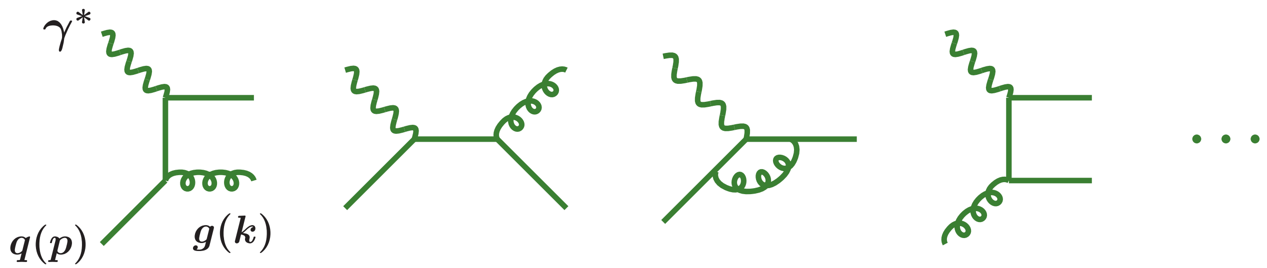

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$C_{a,i}$ **: coefficient functions of F_a**

Γ_{ik} **: $1/\epsilon$ - poles (universal) + $\dots$, $\overline{\text{MS}}$ scheme**

Renormalized parton distributions f_i : splitting functions P_{ij}

$$\frac{\partial}{\partial\ln Q^2}\,f_i \;=\; \frac{\partial\,\Gamma_{ik}}{\partial\ln Q^2}\otimes\Gamma^{-1}_{kj}\otimes f_j \;\equiv\; P_{ij}\otimes f_j$$

Flavour decomposition of the evolution (I)

Quark-gluon and gluon-quark splitting functions: (anti-)flavour independent

$$P_{\mathrm{gq}} \equiv P_{\mathrm{gq}_i} = P_{\mathrm{g\bar{q}}_i} \;, \qquad P_{\mathrm{qg}} \equiv 2n_f P_{\mathrm{q}_i\mathrm{g}} = 2n_f P_{\mathrm{\bar{q}}_i\mathrm{g}}$$

$\Rightarrow$ **quark-(anti-)quark differences** $q_i - q_k$ **and** $q_i - \bar{q}_k$ **decouple from** g

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General structure of the (anti-)quark (anti-)quark splitting functions

$$P_{{\rm q}_i{\rm q}_k} \;=\; P_{{\rm \bar{q}}_i{\rm \bar{q}}_k} \;=\; \delta_{ik} P_{\rm qq}^{\rm v} + P_{\rm qq}^{\rm s}$$

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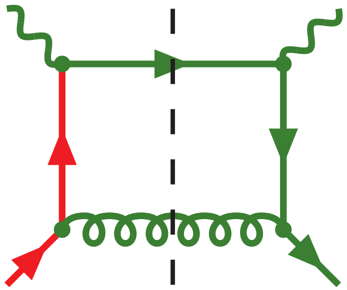

$$P_{\rm qq}^{\rm v} = \mathcal{O}(\alpha_{\rm s})$$

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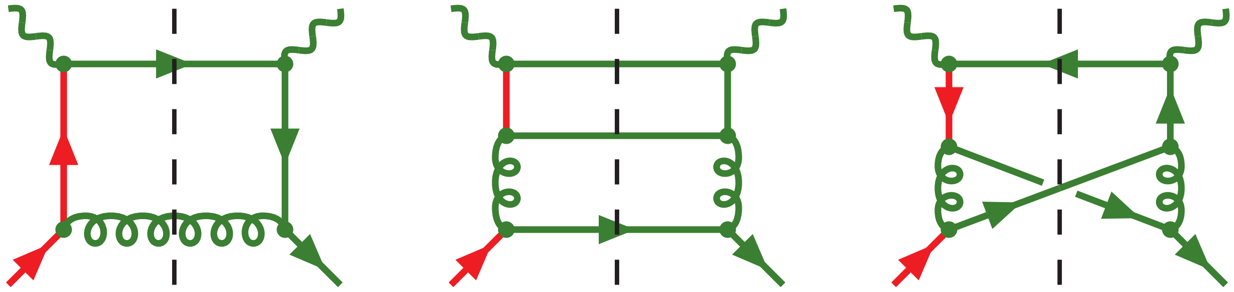

$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ $\qquad$ $P^{\mathrm{s}}_{\mathrm{qq}},\ P^{\mathrm{s}}_{\mathrm{q\bar{q}}} \,:\, \alpha_{\mathrm{s}}^2$ $\qquad$ $P^{\mathrm{v}}_{\mathrm{q\bar{q}}} \,:\, \alpha_{\mathrm{s}}^2$

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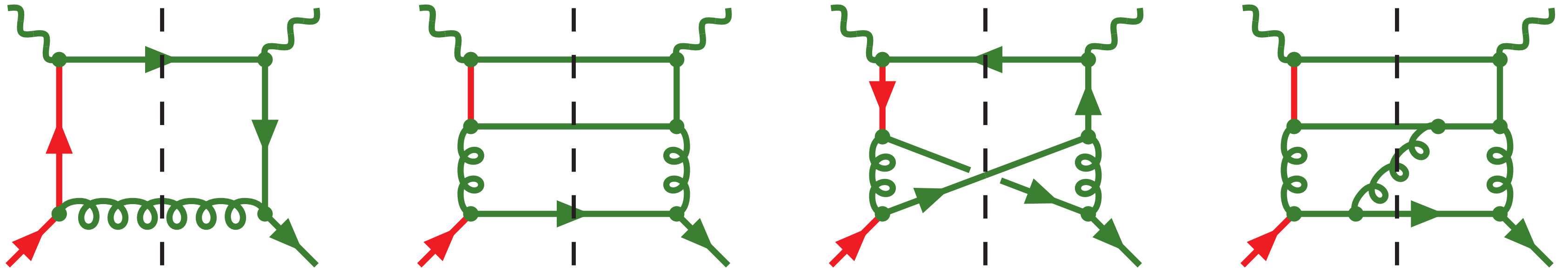

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$\Rightarrow$ **three types of independent difference (non-singlet, ns) combinations:**

Flavour decomposition of the evolution (II)

$2(n_f - 1)$ **flavour asymmetries of** $q_i \pm \bar{q}_i$ **+ one total valence distribution**

$$q^{\pm}_{\mathrm{ns},ik} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \;, \quad q^{\mathrm{v}}_{\mathrm{ns}} = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$

with

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Flavour-singlet quark distribution q_{s} : maximal coupling to g

$$q_{\mathrm{s}} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r) \;, \quad \frac{d}{d \ln \mu^2} \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

with (ps = ‘pure singlet’)

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$$P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{\bar{q}q}}) \equiv P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$$

Evolution: transform quark input to ‘evolution basis’ of q_{s}, $q^{\mathrm{v}}_{\mathrm{ns}}$ **and, e.g.,**

$$v^{\pm}_l = \sum_{i=1}^{k} (q_i \pm \bar{q}_i) - k(q_k \pm \bar{q}_k) \; , \quad k = 2, \ldots, n_f \; , \; l \equiv k^2 - 1 \; ,$$

evolve non-singlet / singlet, and transform back to, say, $u \pm \bar{u}$, $d \pm \bar{d}$ **etc**

Flavour symmetry breaking by evolution

Input $u = u_{\rm v} + \bar{u}$, $d = d_{\rm v} + \bar{d}$ **with SU(2) - symm. sea,** $\bar{u}(\mu_0^2) = \bar{d}(\mu_0^2)$

$\Rightarrow \quad v_3^+ \; = \; u_{\rm v} + 2\bar{u} - d_{\rm v} - 2\bar{d} \; = \; u_{\rm v} - d_{\rm v} \; = \; v_3^-$ **at input scale** μ_0^2

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(Truncated) NLO evolution

$$v_3^-(a_{\rm s}) \;=\; \left\{ 1 + (a_{\rm s} - a_0)\, R_1^- \right\} \left(\frac{a_{\rm s}}{a_0} \right)^{-R_0^{\rm ns}} (u_{\rm v} - d_{\rm v})(a_0)$$

$$v_3^+(a_{\rm s}) \;=\; \left\{ 1 + (a_{\rm s} - a_0)\, R_1^+ \right\} \left(\frac{a_{\rm s}}{a_0} \right)^{-R_0^{\rm ns}} (u_{\rm v} - d_{\rm v})(a_0)$$

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SU(2) sea symmetry not preserved by NLO evolution for $u_{\mathrm{v}} \neq d_{\mathrm{v}}$ **(p)**

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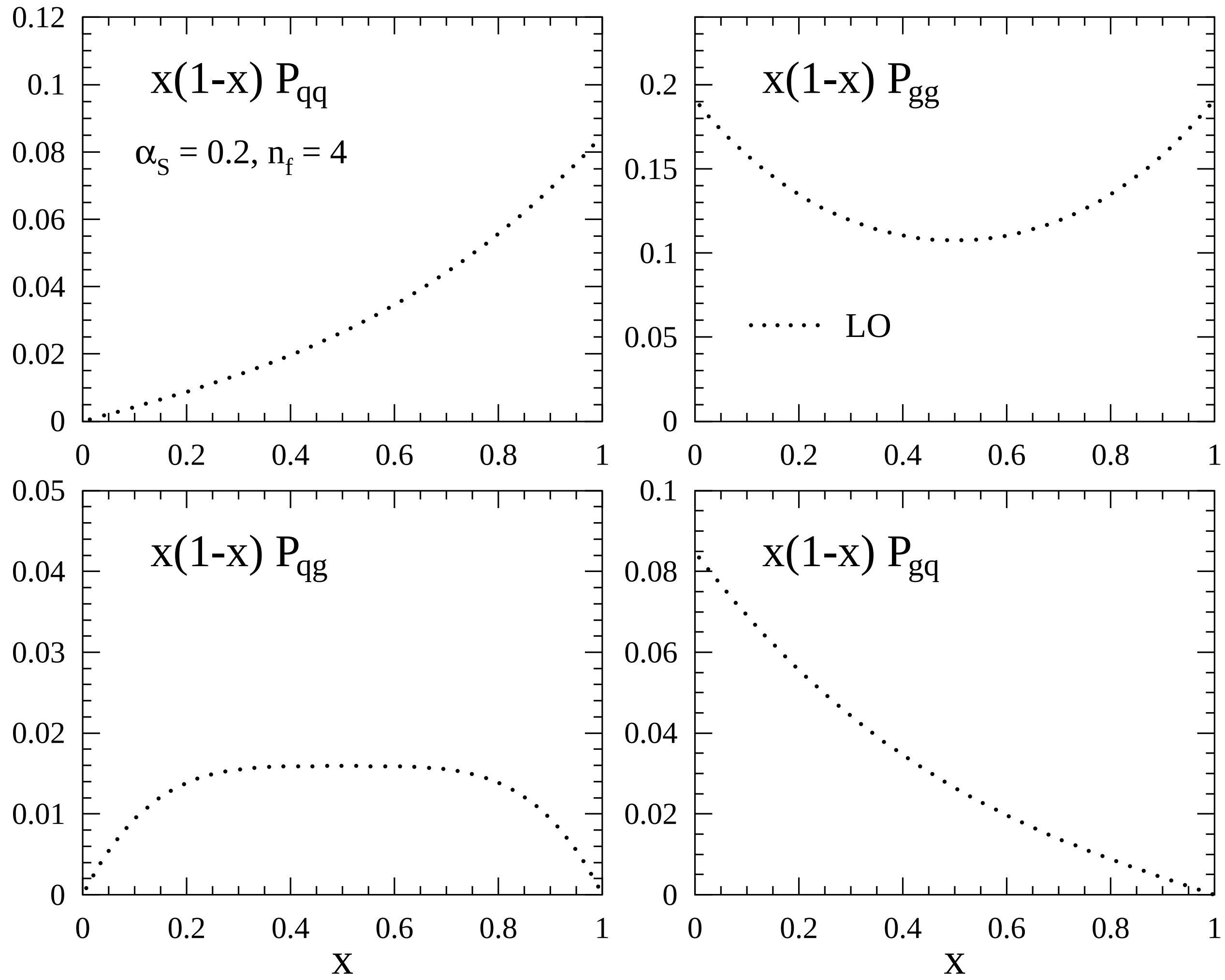
SU(2) sea symmetry not preserved by NLO evolution for $u_\mathrm{v} \neq d_\mathrm{v}$ (p)

Analogous situation: $s \neq \bar{s}$ at NNLO even for $(s - \bar{s})(\mu_0^2) = 0$:

small effects, 'dynamical' $s - \bar{s}$ looked at for $\sin^2\theta_{\mathrm{weak}}$ from $\nu/\bar{\nu}$ DIS

Singlet splitting functions $P(x<1)$ to NNLO



Note: Plotted information incomplete for (math.) distributions P_{qq} and P_{gg}

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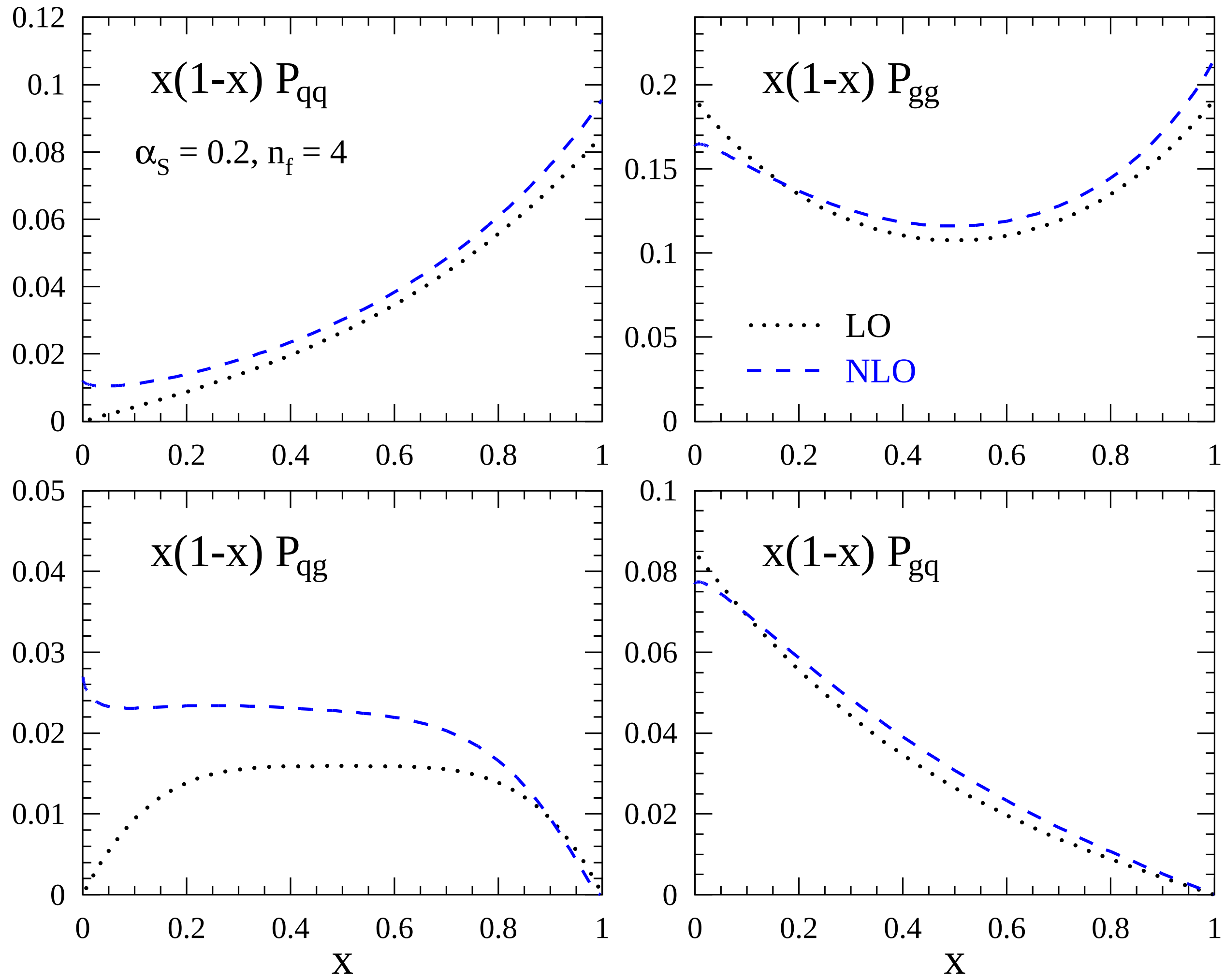

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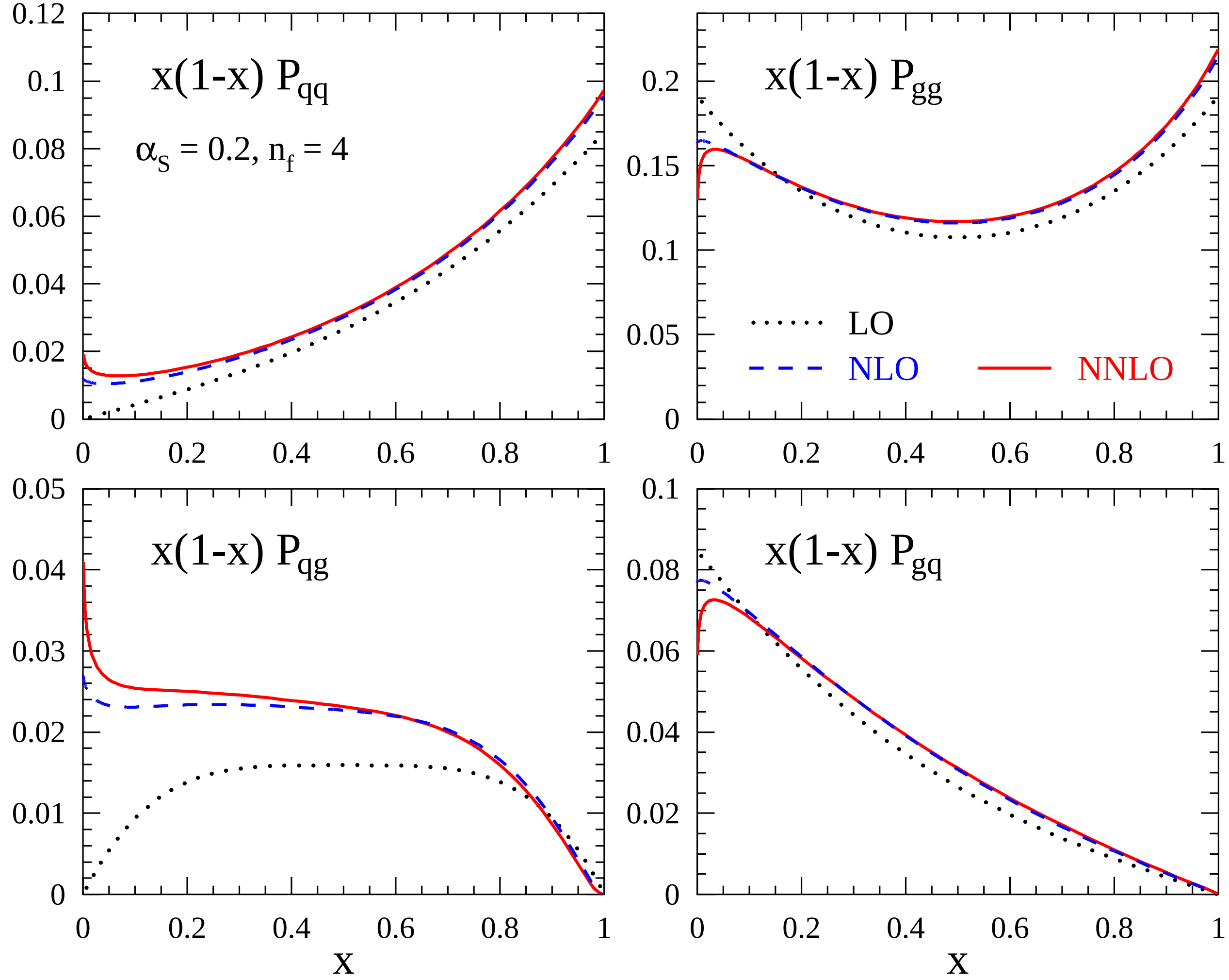

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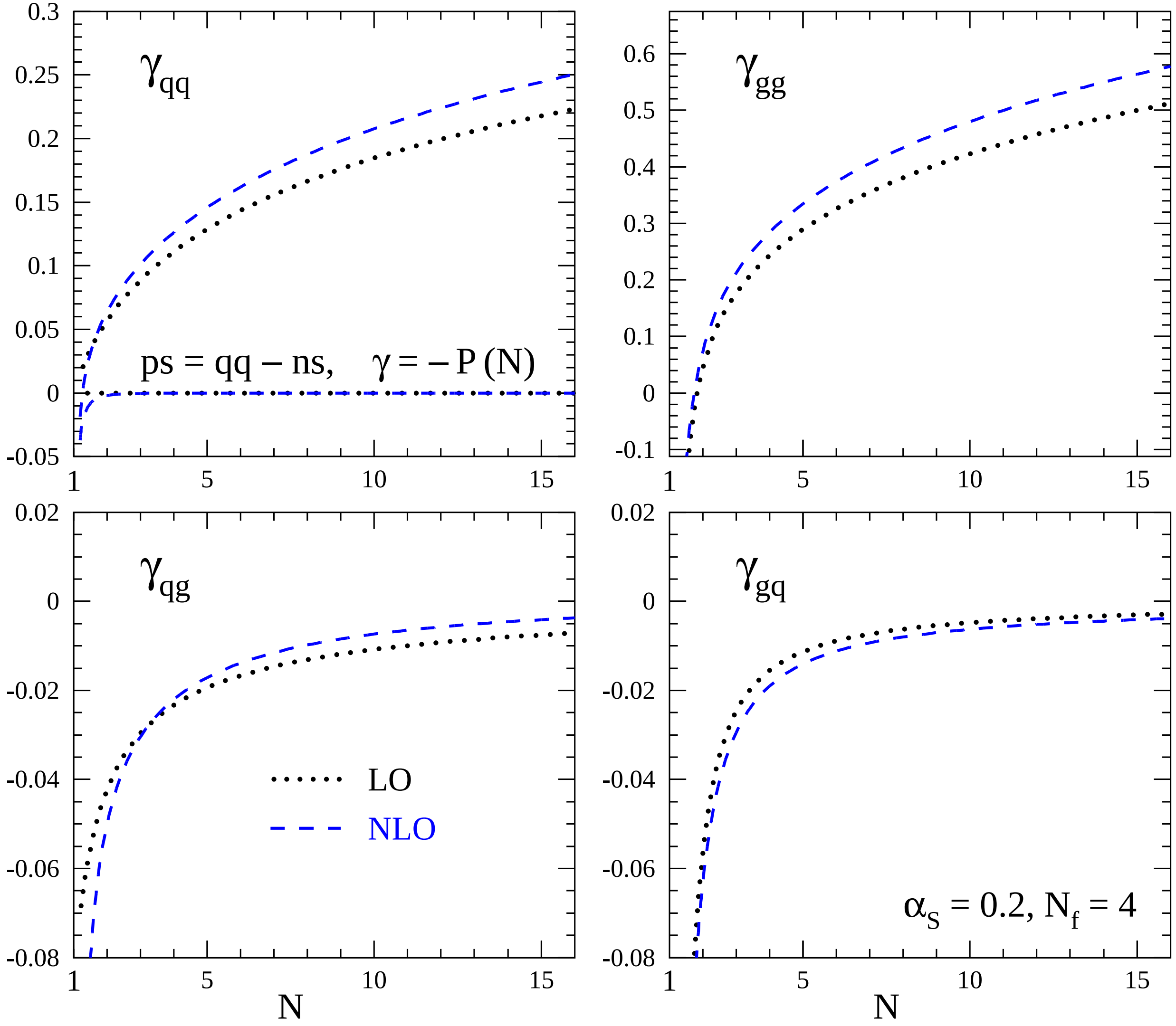
Stability of the perturbative expansion an issue only at small values of x

Mellin-N space splitting functions to NNLO



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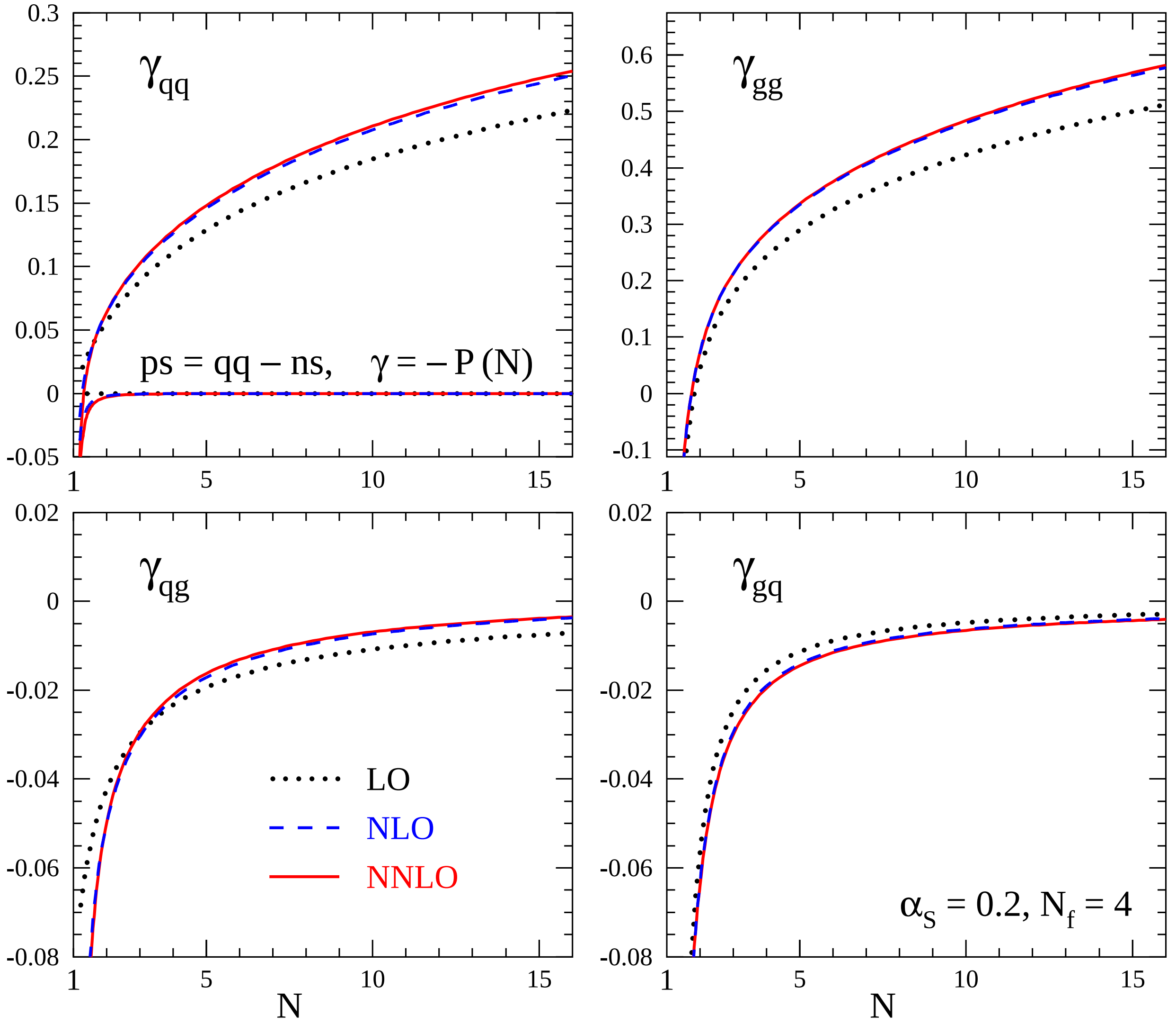
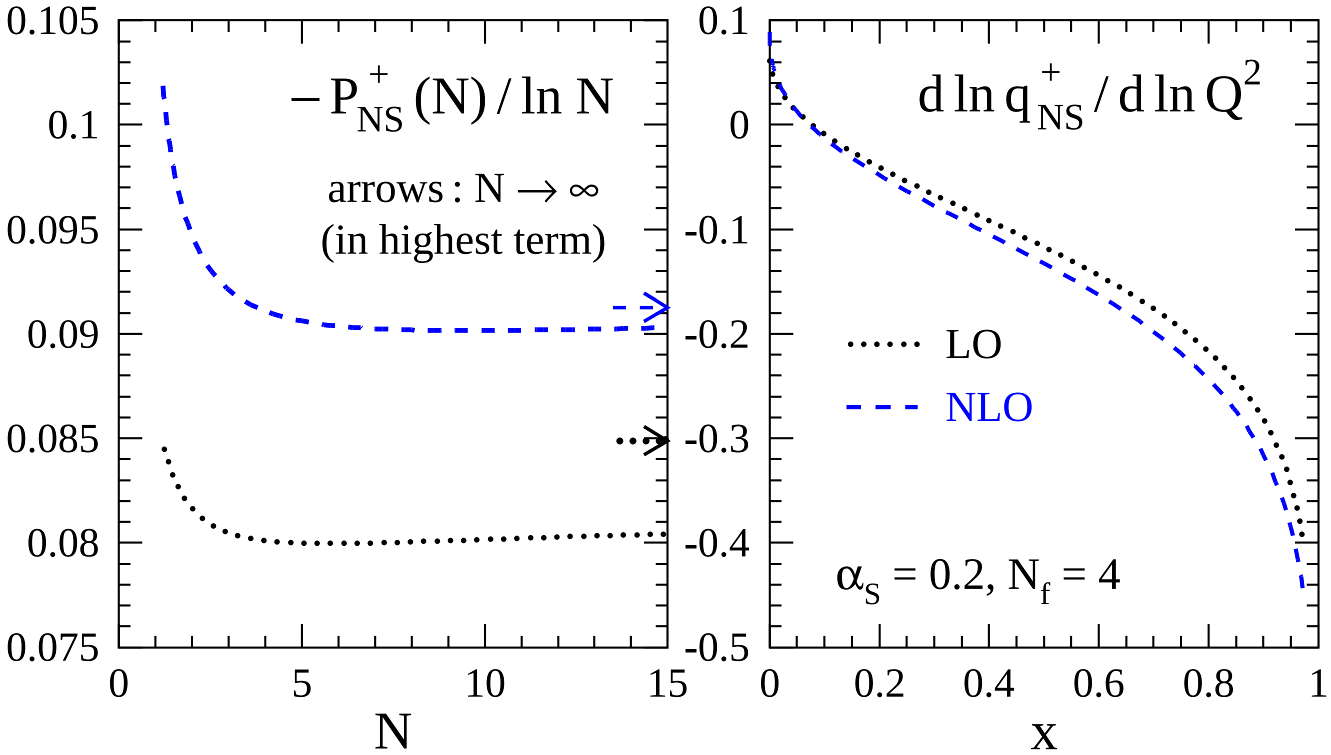


$N > 2$: **off-diagonal (NNLO to about 5%) $\ll$ diagonal (NNLO to about 2%)**

$\overline{\text{MS}}$ non-singlet evolution at large N / large x

Moments: $A^N = \int_0^1 dx \, x^{N-1} A(x)\,, \quad (1-x)_+^{-1} \leftrightarrow \ln N + \gamma_e + \mathcal{O}(1/N)$



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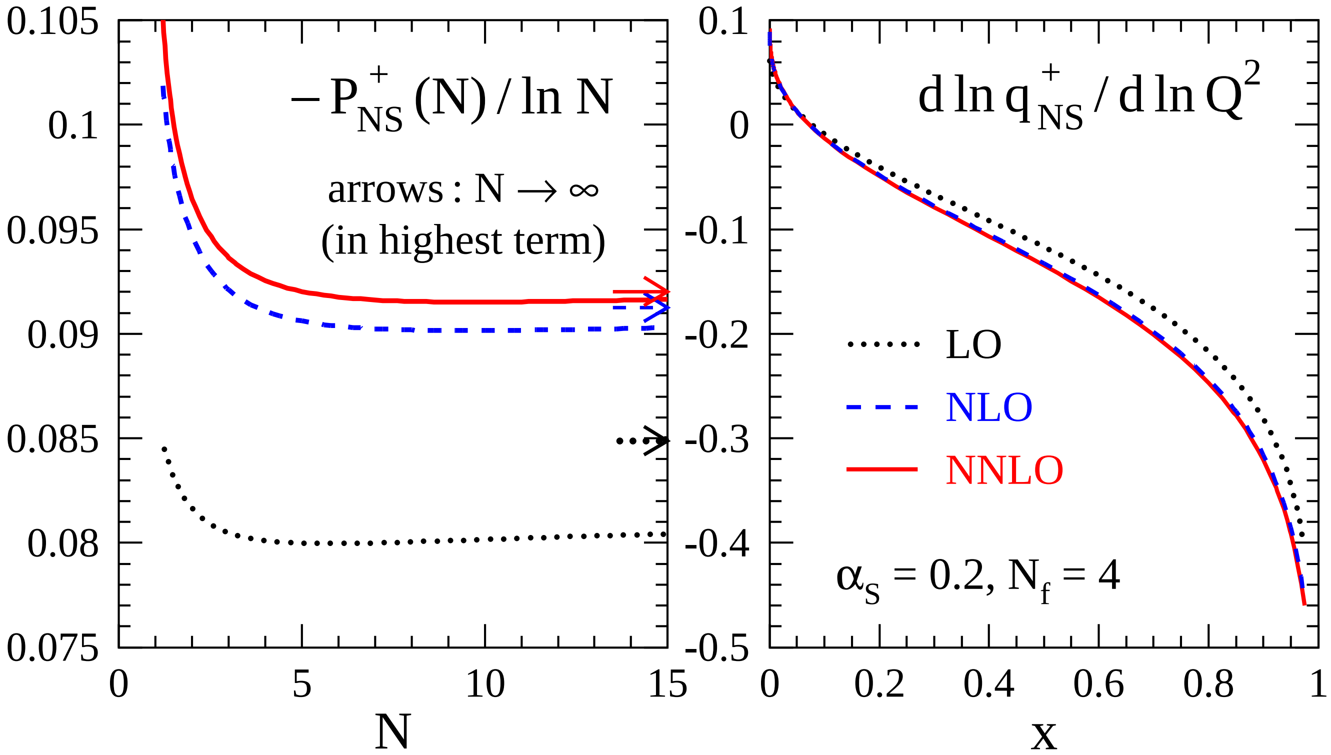

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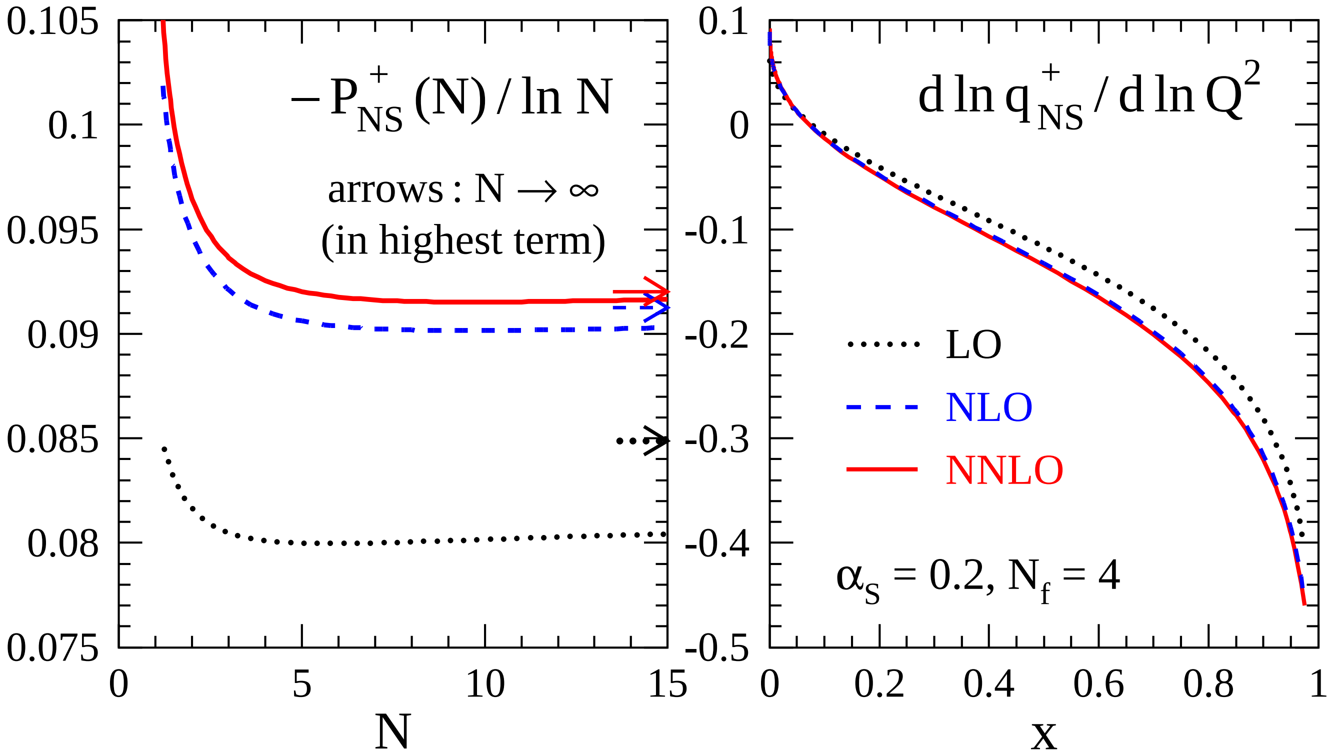

N³LO: P^+_{ns} **computed for** $N = 2\,,\ n_f = 3$ **Baikov, Chetyrkin (06)**

$$P^+_{\text{ns}} = -0.283\,\alpha_s\,[\,1 + 0.869\,\alpha_s + 0.798\,\alpha_s^2 + 0.926\,\alpha_s^3 + \ldots]$$

$N > 2\,,\ n_f > 3$ **: similar / smaller** $\ln N$ **coeff's.** $\simeq$ **1% accuracy at** $\alpha_s \lesssim 0.25$

Small-x behaviour of the splitting functions

NNLO non-singlet : $P^{(2)i}_{x\to 0}(x) \;=\; D^{\,i}_0 \ln^4 x \;+\; \ldots \;+\; D^{\,i}_3 \ln x \;+\; \mathcal{O}(1)$

Generally terms up to $\ln^{2k} x$ at order $\alpha_{\rm s}^{\,k+1}$

$D^{\,i}_0$: Blümlein, A.V. (95)

Coefficients for 'plus' case, like $u+\bar{u}-(d+\bar{d}\,)$ for $n_f = 4$ with $a_{\rm s} = \dfrac{\alpha_{\rm s}}{4\pi}$

$$D^{+}_0 \cong 1.580\,, \quad D^{+}_1 \cong 20.18\,, \quad D^{+}_2 \cong 175.3\,, \quad D^{+}_3 \cong 720.3$$

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NNLO singlet : $$P^{(2)}_{\mathrm{ab},x\to 0}(x) \;=\; E^{\mathrm{ab}}_1 \frac{\ln x}{x} \;+\; E^{\mathrm{ab}}_2 \frac{1}{x} \;+\; \mathcal{O}(\ln^4 x)$$

Generally terms up to $x^{-1}\ln^k x$ (gb) and $x^{-1}\ln^{k-1} x$ (qb) at order α_s^{k+1}

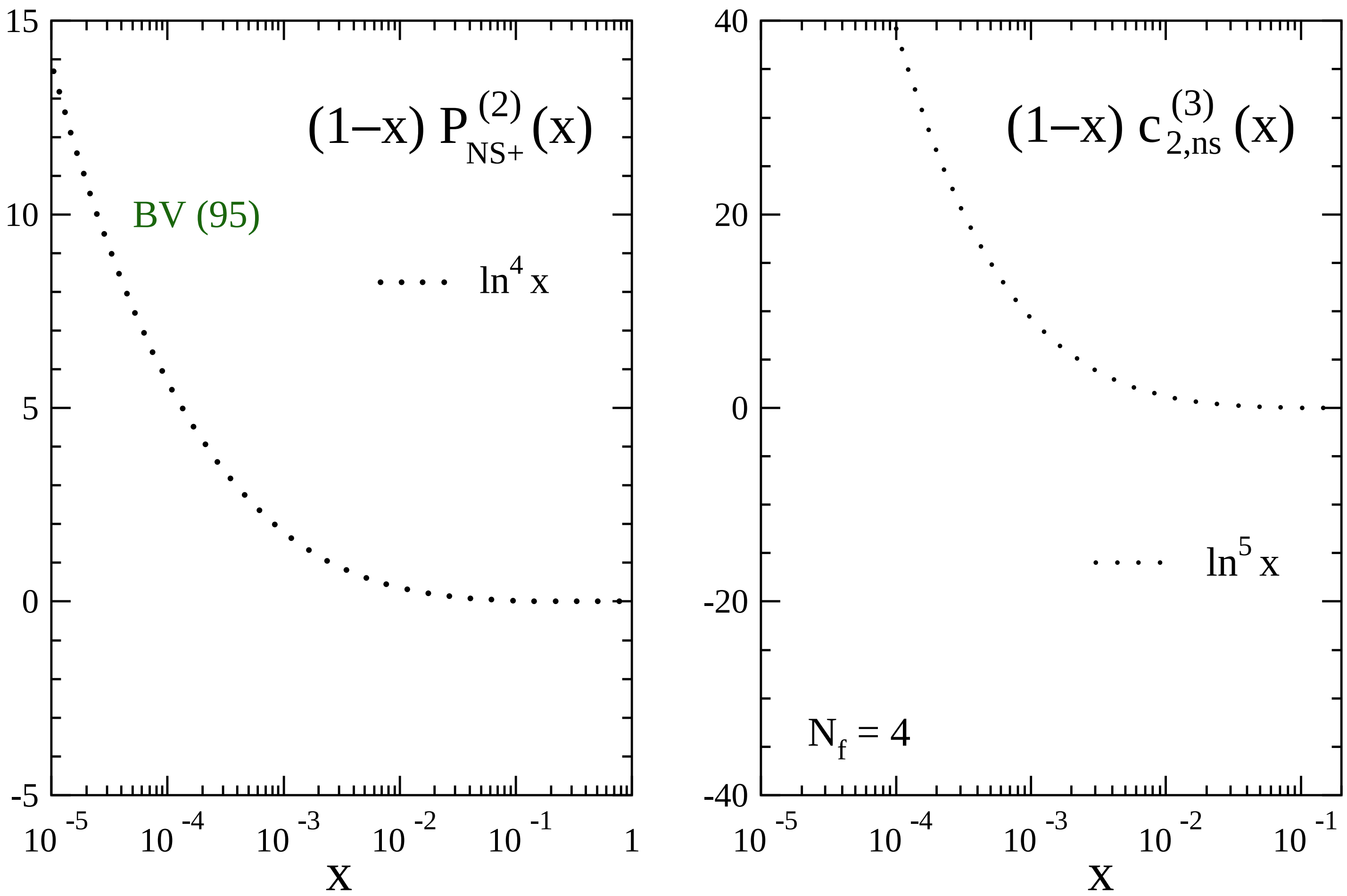
E^{qb}_1 : Catani, Hautmann (94), E^{gg}_1 : Fadin, Lipatov (98)

$n_f = 4$:
$$E^{\mathrm{qg}}_1 \cong -1194.7 \;\;,\quad E^{\mathrm{qg}}_2 = -4999.9$$
$$E^{\mathrm{gg}}_1 \cong +3304.9 \;\;,\quad E^{\mathrm{gg}}_2 = +14901$$

More often than not, **Large** logarithms have small coefficients

Non-singlet three-loop quantities at small x

Order α_s^n: small-x 'double logs' $\ln^{2k} x$ with $k \leq n-1$ $(n-\frac{1}{2})$ in P (c)



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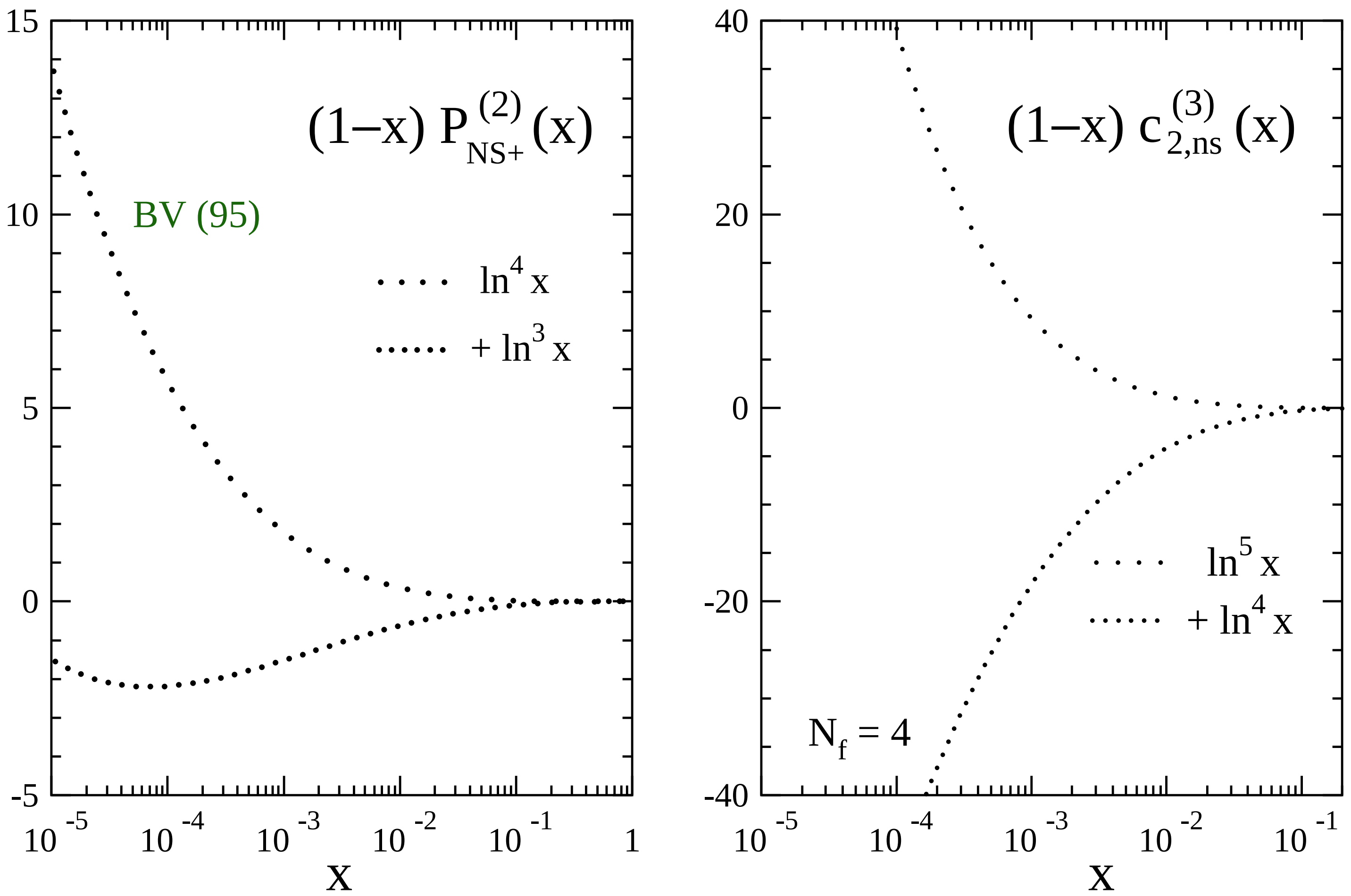

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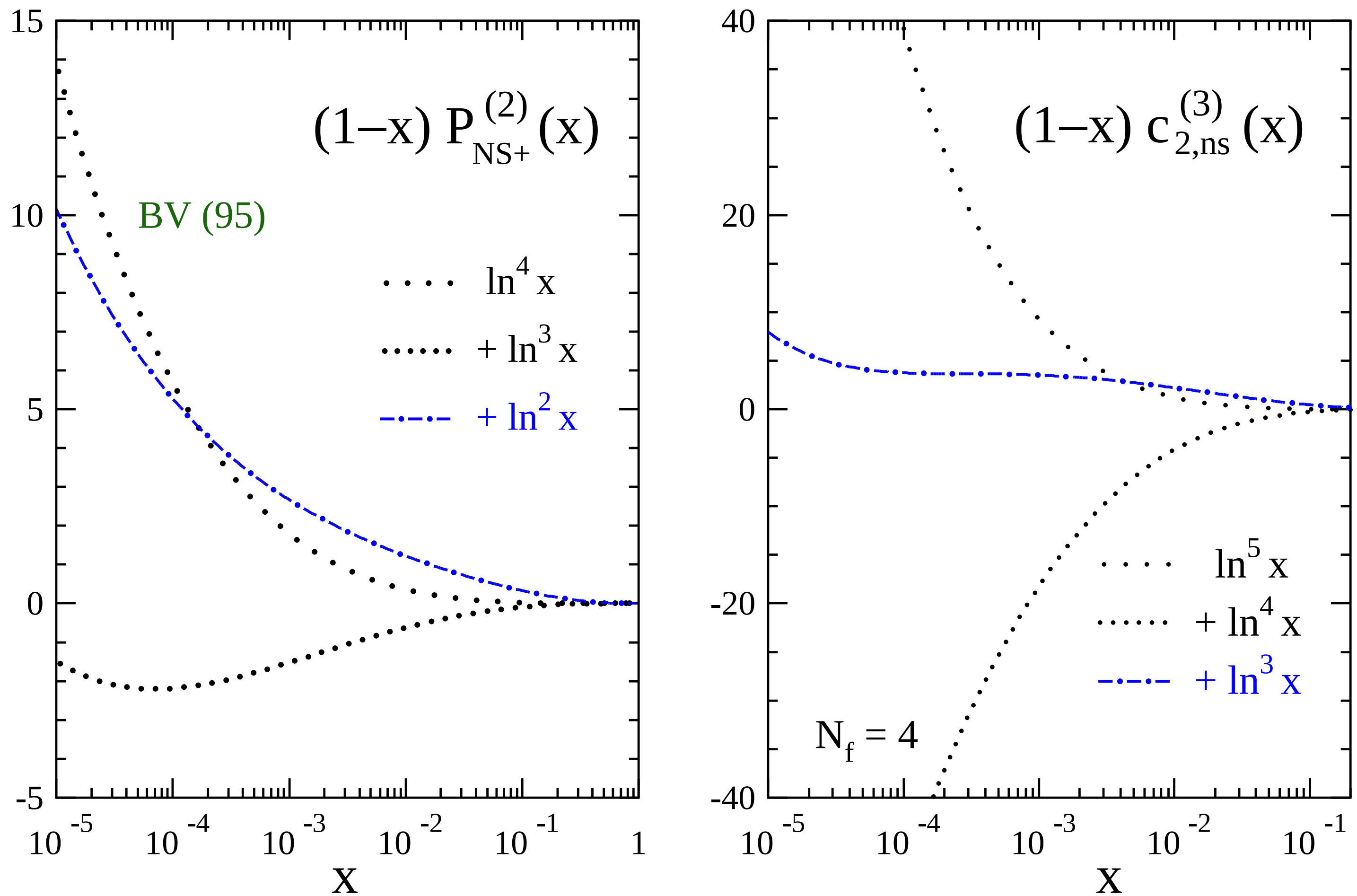

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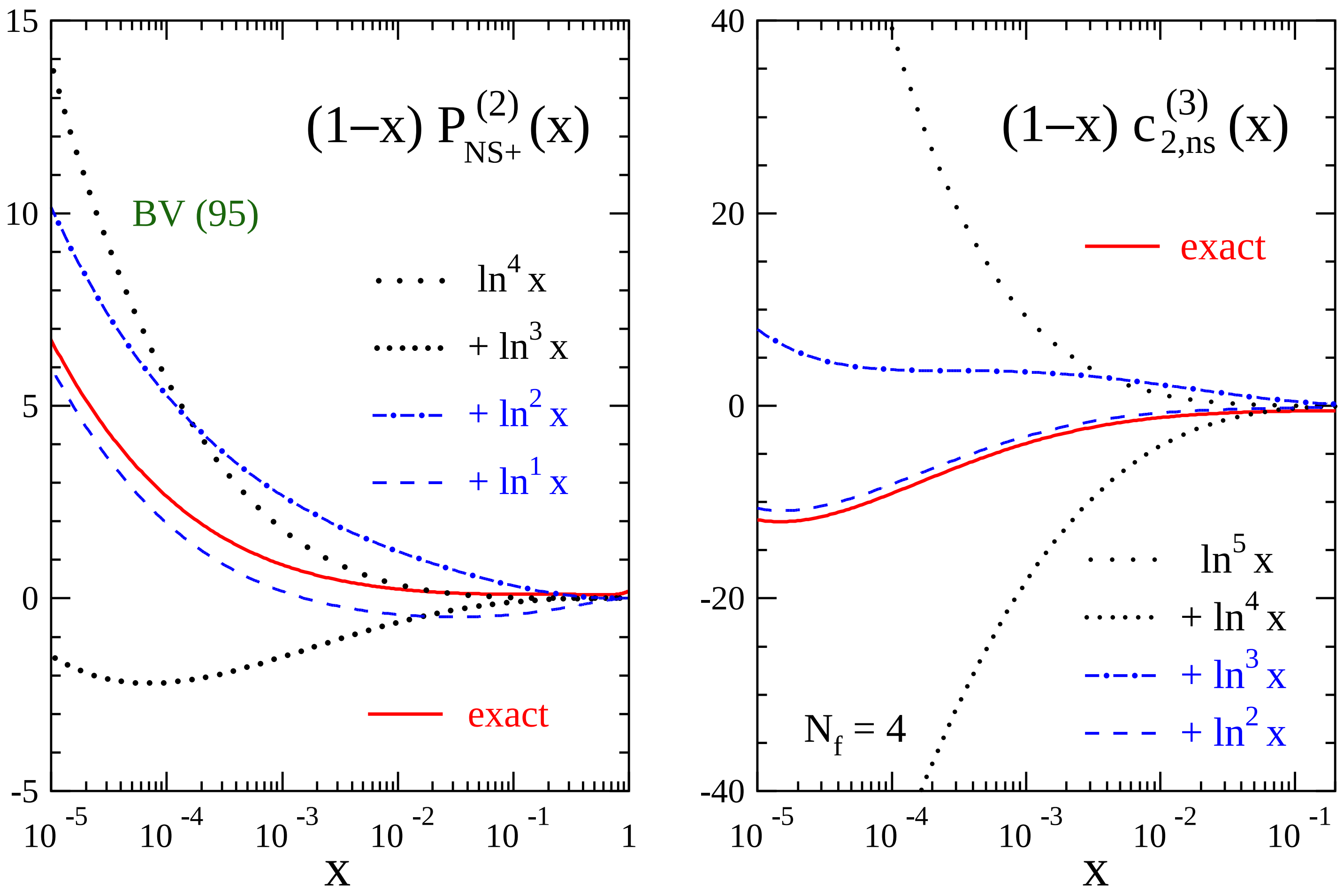
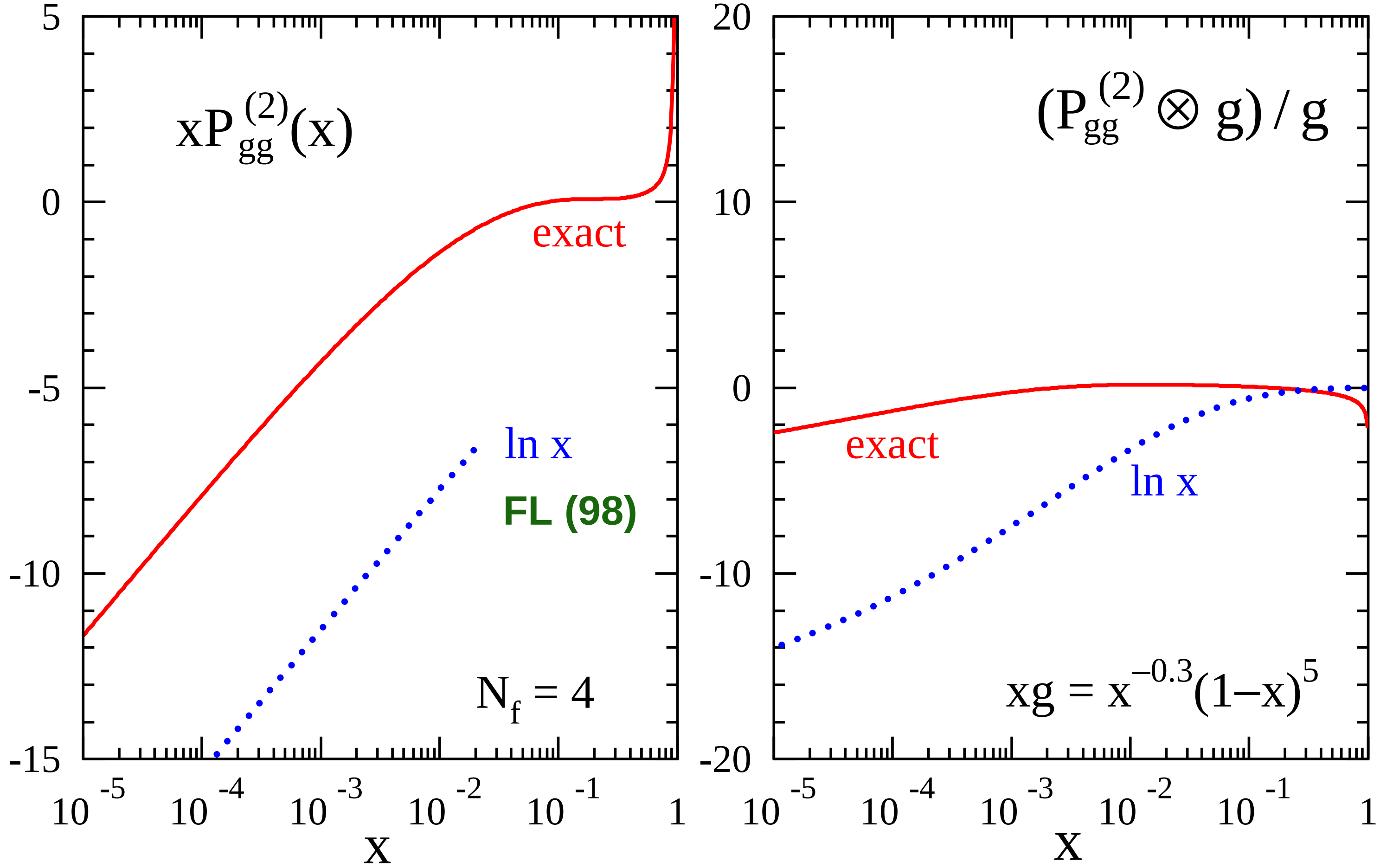


x-values for colliders: not even shape guaranteed by (next-to-) leading logs

Singlet splitting and evolution at small x

Splitting functions $\longrightarrow$ observables: **Mellin convolutions** $\int_x^1 \frac{dy}{y} P(y)\, f\left(\frac{x}{y}\right)$



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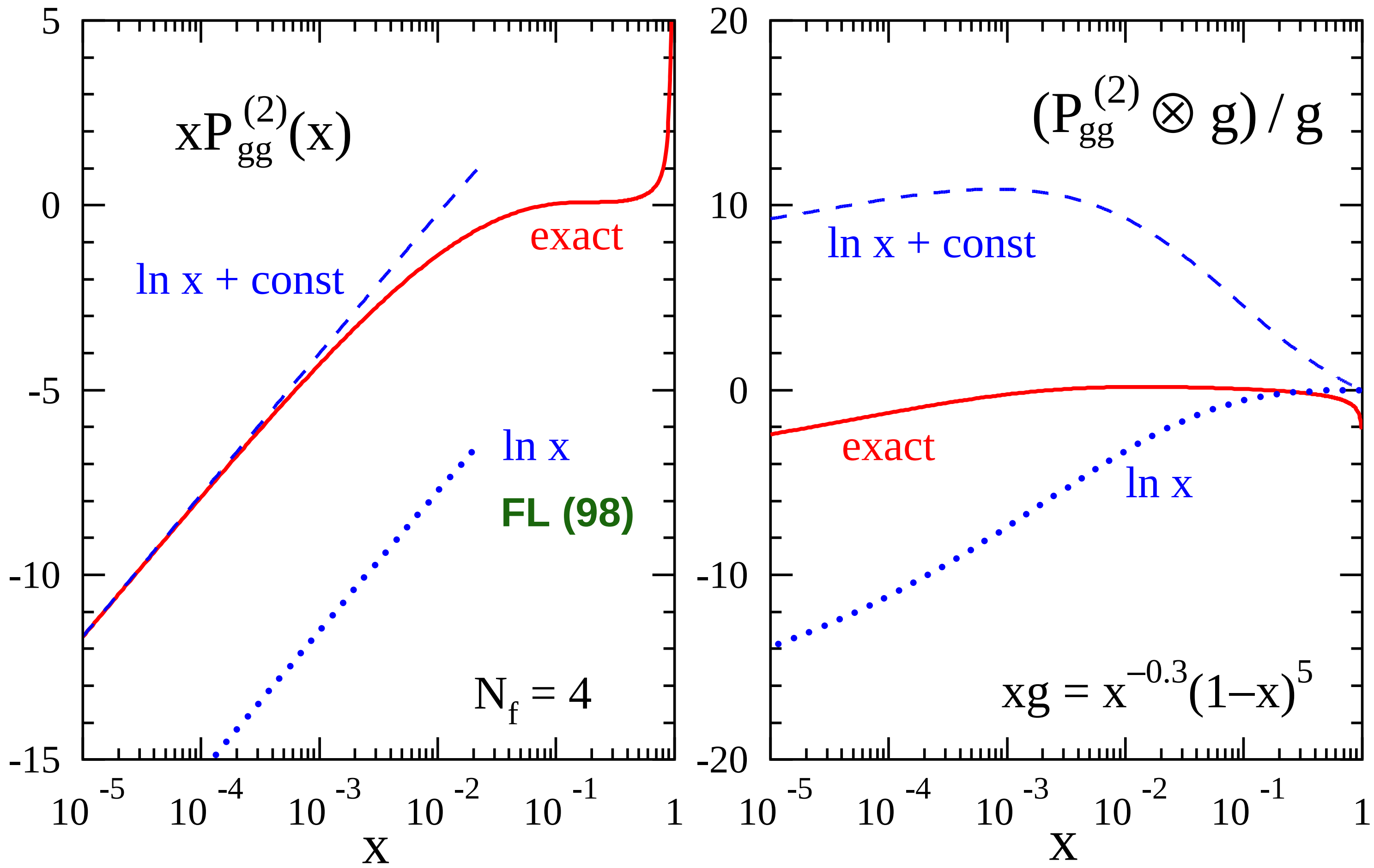
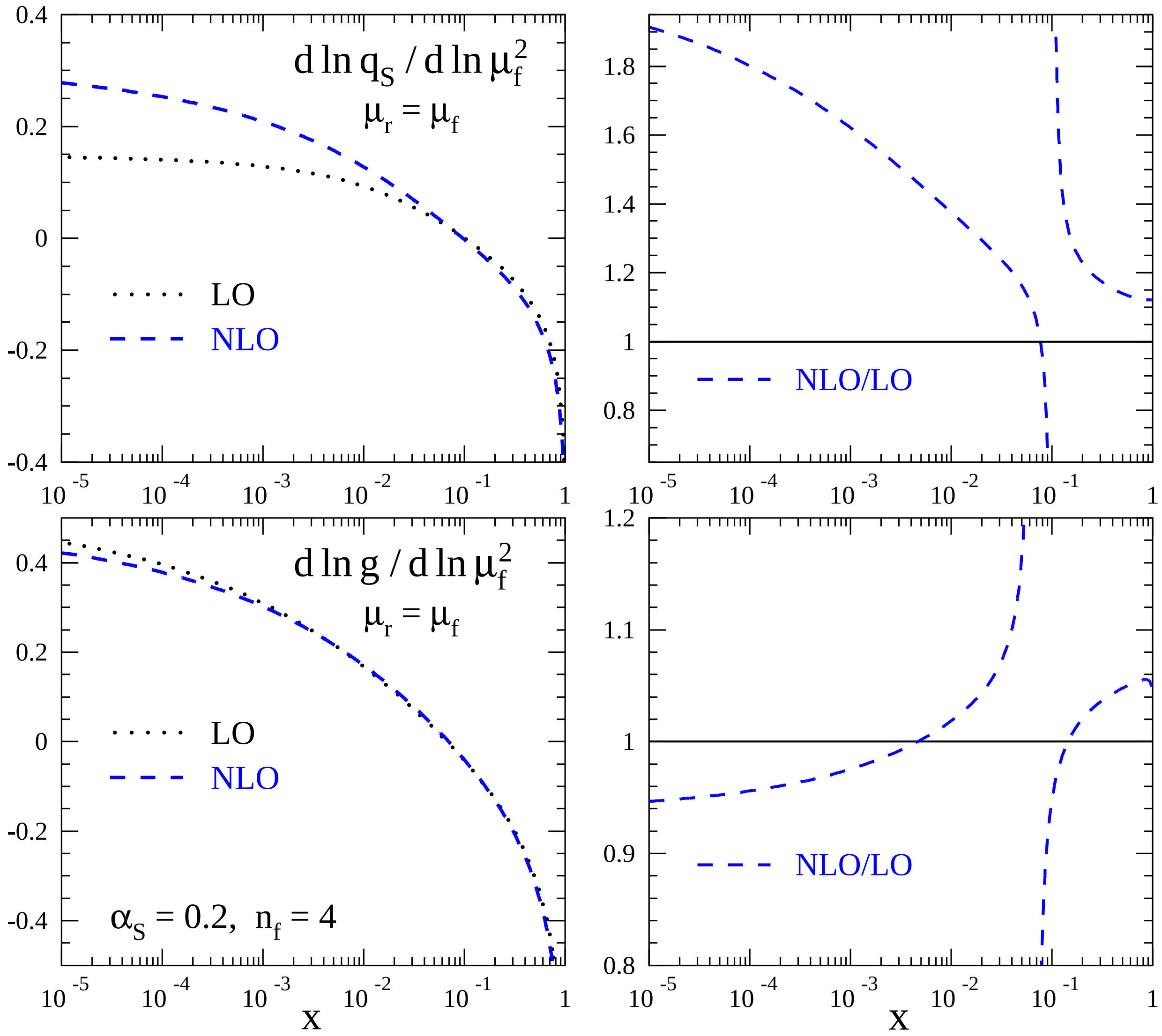


General: small-x limits of pQCD functions insufficient due to convolutions

Scale derivatives of singlet parton densities



scale $\approx$ 30 GeV^2

quark distrib'n

gluon distrib'n

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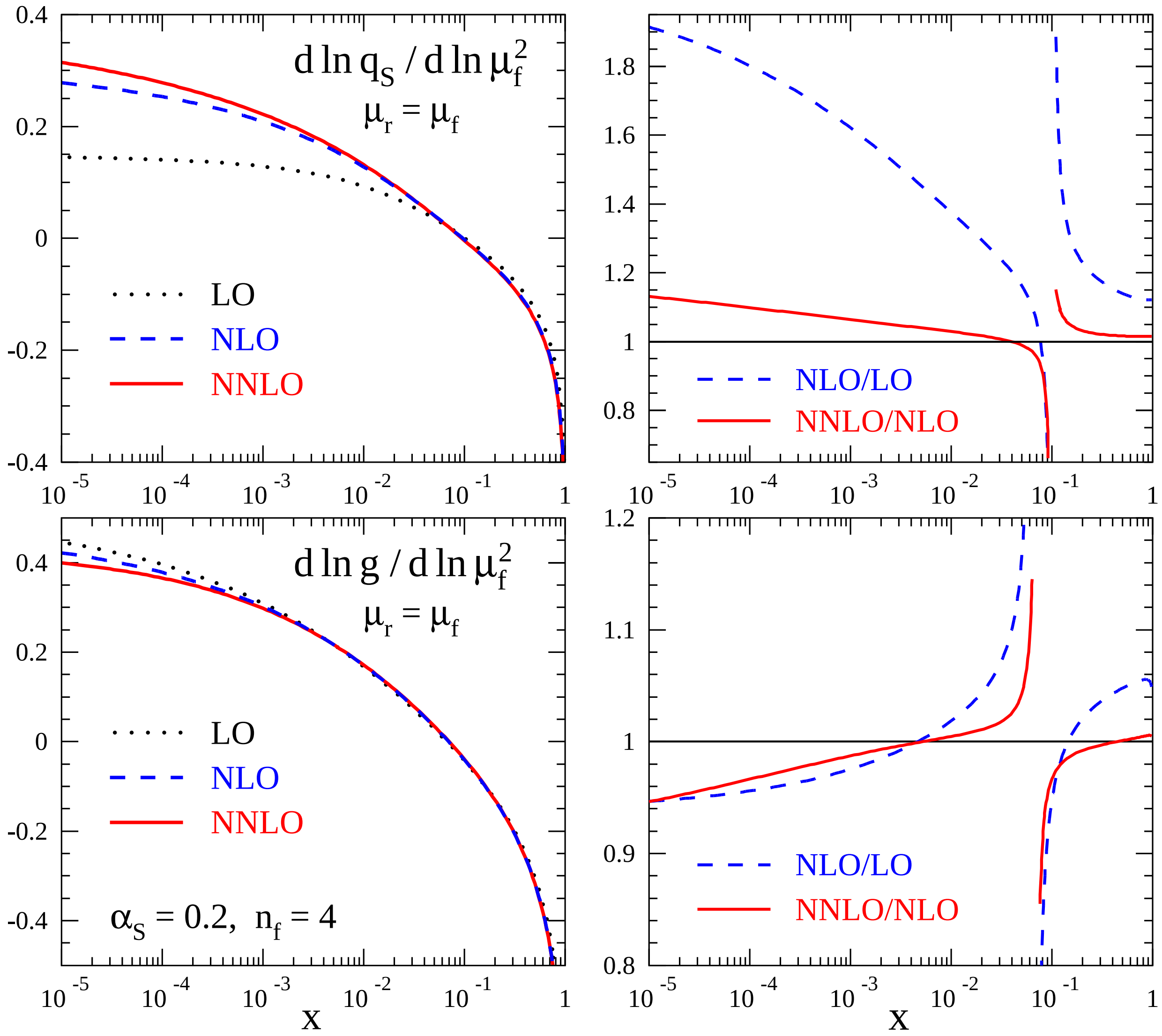

scale ≈ 30 GeV^2

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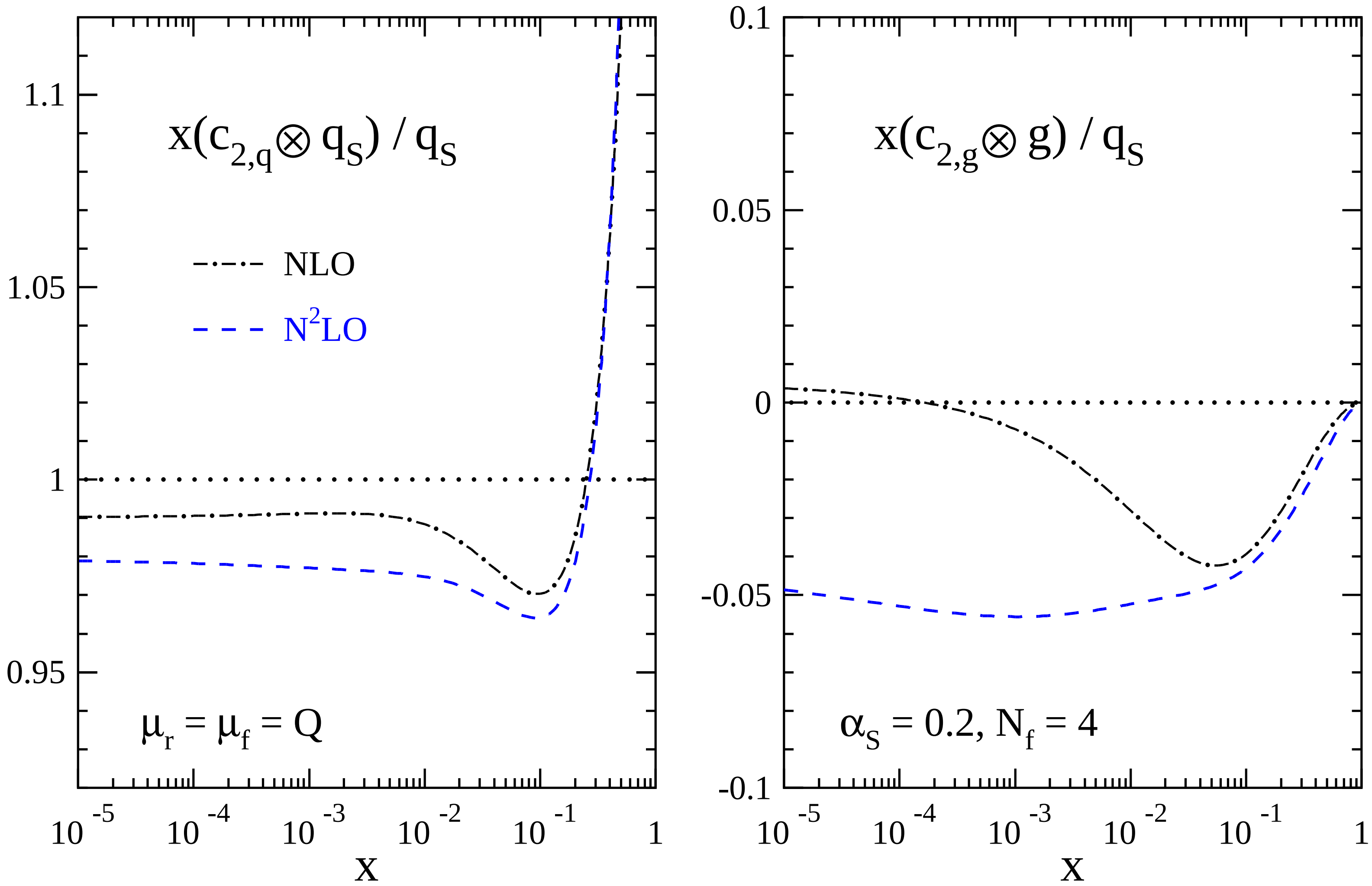
Good convergence at collider-x – but NNLO is 10% for q_s at $x = 10^{-4}$

Expansion of photon-exchange F_2 to N^{3}LO



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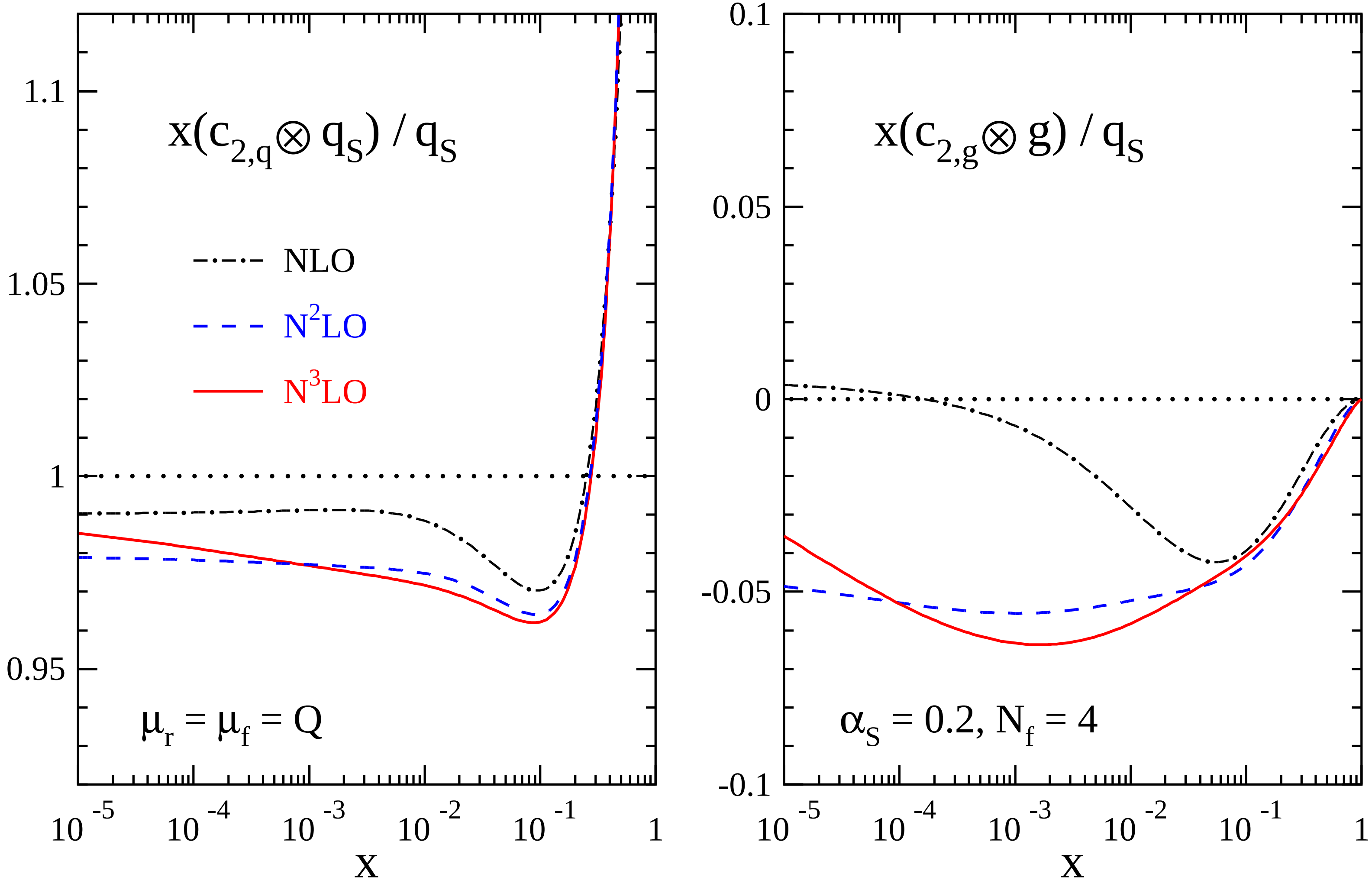
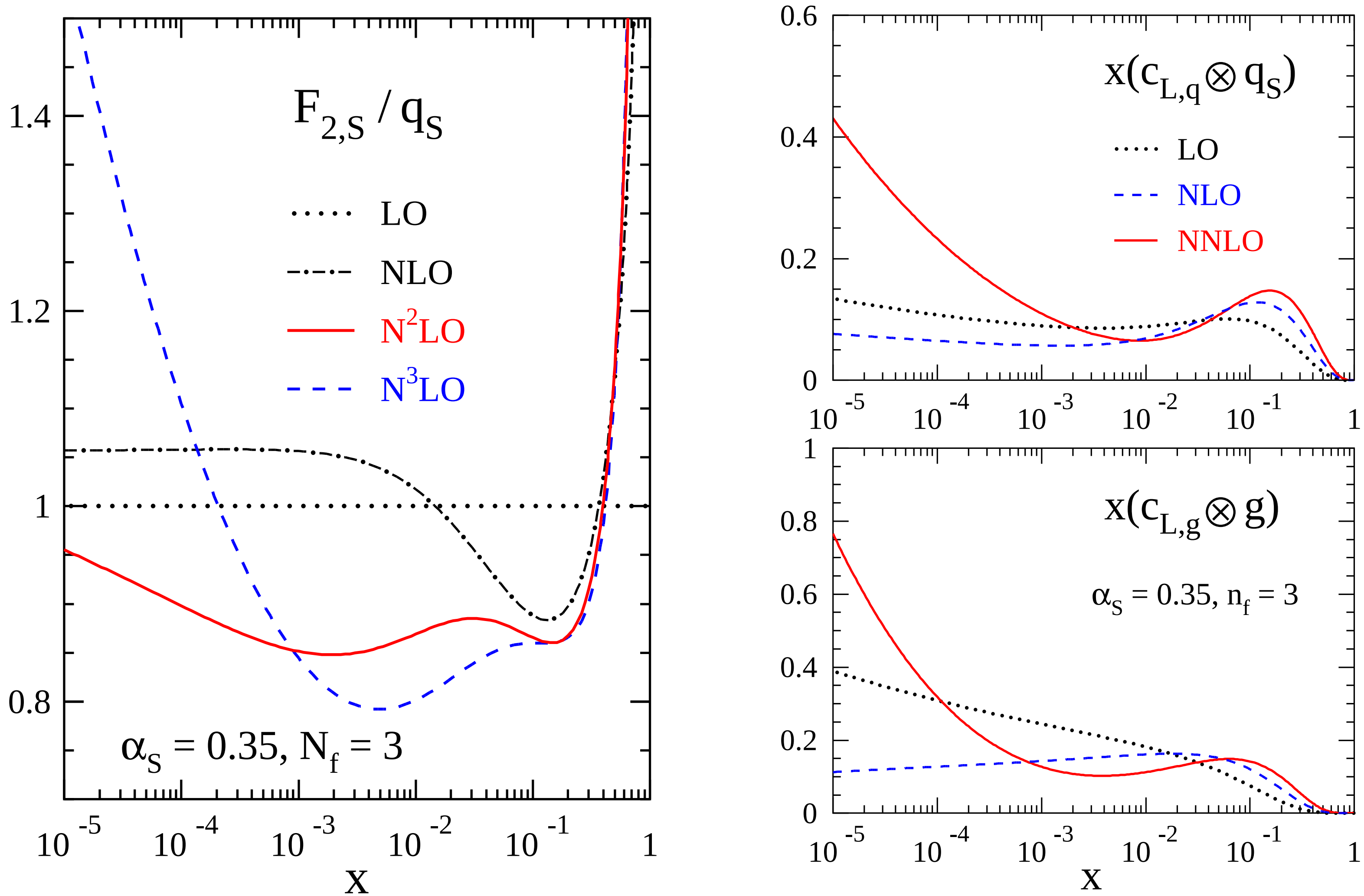


Total N^{3}LO corr. $\leq 1\%$ at $4 \cdot 10^{-5} \leq x \leq 0.65$. N^{3}LO > NNLO for $x \lesssim 10^{-8}$

Disclaimer (II): beware of small x at low Q^2



$Q^2 \approx 2$ GeV^2: expansion out of control at $x \lesssim 10^{-4}$ (F_2) and $x \lesssim 10^{-3}$ (F_L)

Available evolution codes including NNLO

x**-space: discretization in** x, $\mu_{\rm f}$ **of coupled integro-differential equations**

HOPPET (G. Salam, publ. 2008), http://hepforge.cedar.ac.uk/hoppet/ with J. Rojo

QCDNUM (M. Botje, now v.17β**), http://www.nikhef.nl/˜h24/qcdnum/**

N**-space: ordinary diff. eqs., time-ordered exponential,** $N \to x$ **numerical**

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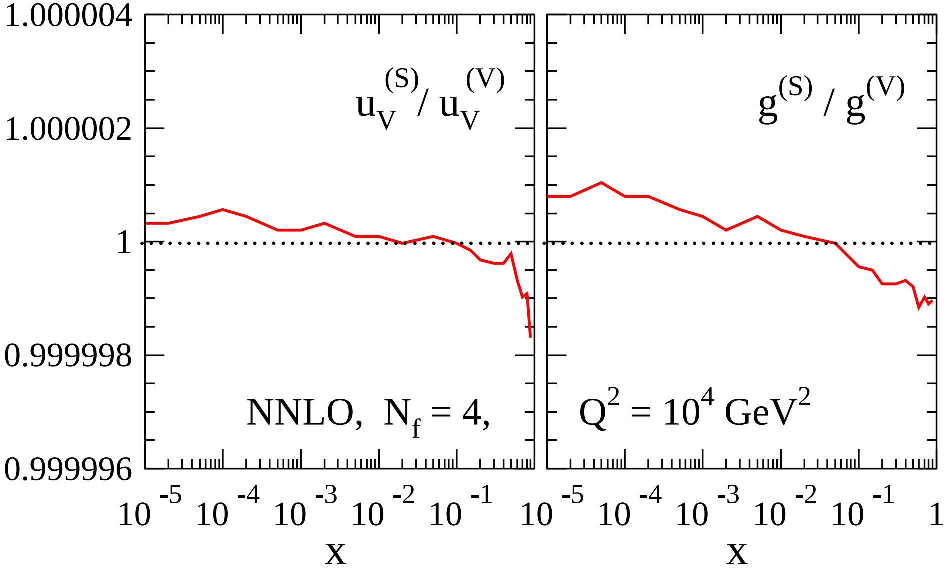
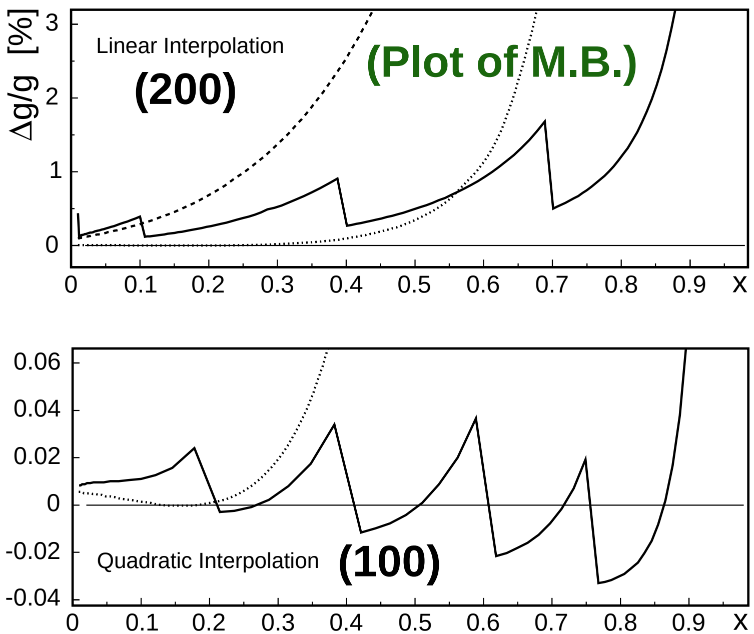
Sample comparisons





Benchmark tables for the parton evolution

Evolution of Les Houches (2001) reference input at scale $\mu^2_{\mathrm{f},0} = 2$ GeV^2

$$
\begin{aligned}
xu_v(x, \mu^2_{\mathrm{f},0}) &= 5.1072\ x^{0.8}\ (1-x)^3 \quad , \quad \ldots \\
xg\,(x, \mu^2_{\mathrm{f},0}) &= 1.7000\, x^{-0.1}\,(1-x)^5
\end{aligned}
$$

with

$$\alpha_s(\mu^2_r = 2\ \text{GeV}^2) = 0.35$$

at LO, NLO and NNLO, for $\mu_r = \{0.5,\ 1,\ 2\}\,\mu_f$, with fixed and variable N_f

Use of two completely different codes.

G. Salam, A.V. (2002, 05)

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Use of two completely different codes. G. Salam, A.V. (2002, 05)

Five-digit agreement over wide range in x, μ_f^2 $\Rightarrow$ reference tables

Example: (iterated) NNLO results, $\mu_r = 2\mu_f$, $N_f = 4$ at $\mu_f^2 = 10^4$ GeV^2

$$
\begin{aligned}
x = 10^{-5}\ , &\quad xu_v = 2.9032\cdot 10^{-3}\ , \quad \ldots\ , \quad xg = 2.2307\cdot 10^{2} \\
\ldots x = 0.9\ , &\quad xu_v = 3.6527\cdot 10^{-4}\ , \quad \ldots\ , \quad xg = 1.2489\cdot 10^{-6}
\end{aligned}
$$

Full tables in hep-ph/0204316 (Les Houches), hep-ph/0511119 (HERA-LHC)

Heavy quarks in hard proton processes

$$m_u,\, m_d \ll \Lambda_{\rm QCD}\ ,\quad m_s \lesssim \Lambda_{\rm QCD}$$

Can neglect 'light quark' masses in description of hard proton processes

$\Rightarrow$ **mass singularities, scale-dependent u, d, s, g parton distributions**

Heavy quarks, $m_{c,b,t} \gg \Lambda_{\rm QCD}$: no mass sing., no (evolving) partons distr.

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Example: structure function $F_2^{c\bar{c}}$, disregarding 'intrinsic charm' ($\Leftarrow$ HERA)

$Q \not\gg m_c$ **: u, d, s, g partons + massive charm-prod. coefficient functions**

Fixed flavour-number scheme, FFNS

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$Q \ggg m_c$: terms $m_c/Q \rightarrow 0$, $n_f = 4$ pdf’s (matching), $m = 0$ coeff. fct’s

Zero-mass variable flavour-number scheme, ZM-VFNS

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Zero-mass variable flavour-number scheme, ZM-VFNS

$Q \gg m_c$: terms $m_c/Q \neq 0$, but quasi-collinear logs $\ln(Q/m_c)$ large,

$n_f = 4$ pdf's, 'interpolating' coeff. functions ($\Leftarrow$ prescriptions)

(General-mass) variable flavour-number scheme, (GM-)VFNS

Heavy quarks in the evolution of PDFs and α_s

Here: disregard 'intrinsic charm/bottom' – might be relevant at large x

cf. Pumplin, Lai, Tung (07)

$\overline{\text{MS}}$ evolution of parton densities and α_s with variable number n_f of flavours: matching of effective theories.

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$$
\begin{aligned}
l_i^{(n_f+1)} &= l_i^{(n_f)} + \theta_{m2}\, a_s^2\, A_{qq,h}^{ns,(2)} \otimes l_i^{(n_f)} + \ldots \\
g^{(n_f+1)} &= g^{(n_f)} + \theta_{m2}\, a_s^2 \left[A_{gq,h}^{s,(2)} \otimes q_s^{(n_f)} + A_{gg,h}^{s,(2)} \otimes g^{(n_f)} \right] + \ldots \\
(h+\bar{h})^{(n_f+1)} &= \theta_{m2}\, a_s^2 \left[A_{hq}^{s,(2)} \otimes q_s^{(n_f)} + A_{hg}^{s,(2)} \otimes g^{(n_f)} \right] + \ldots
\end{aligned}
$$

Buza et al. (96), [Bierenbaum, Blümlein, Klein (07)]

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$$
\begin{aligned}
l_i^{(n_f+1)} &= l_i^{(n_f)} + \theta_{m2}\, a_\mathrm{s}^2\, A^{\mathrm{ns},(2)}_{\mathrm{qq,h}} \otimes l_i^{(n_f)} + \ldots \\
g^{(n_f+1)} &= g^{(n_f)} + \theta_{m2}\, a_\mathrm{s}^2 \left[A^{\mathrm{s},(2)}_{\mathrm{gq,h}} \otimes q_\mathrm{s}^{(n_f)} + A^{\mathrm{s},(2)}_{\mathrm{gg,h}} \otimes g^{(n_f)} \right] + \ldots \\
(h+\bar{h})^{(n_f+1)} &= \theta_{m2}\, a_\mathrm{s}^2 \left[A^{\mathrm{s},(2)}_{\mathrm{hq}} \otimes q_\mathrm{s}^{(n_f)} + A^{\mathrm{s},(2)}_{\mathrm{hg}} \otimes g^{(n_f)} \right] + \ldots
\end{aligned}
$$

Buza et al. (96), [Bierenbaum, Blümlein, Klein (07)]

Corresponding N^{m}LO relation for the coupling constant at $\mu_\mathrm{r} = m_h$

$$
a_\mathrm{s}^{(n_f+1)}(m_h^2) = a_\mathrm{s}^{(n_f)}(m_h^2) + \sum_{n=1}^{m} c_n \left(a_\mathrm{s}^{(n_f)}(m_h^2) \right)^{n+1}
$$

Known to N^{3}LO: $c_1 = 0\,,\ c_{2,3} \neq 0$ Chetyrkin, Kniehl, Steinhauser (97)

Recent fits of (N)NLO parton distributions

Global fits: DIS, fixed-target Drell-Yan, W/Z and jets at the Tevatron

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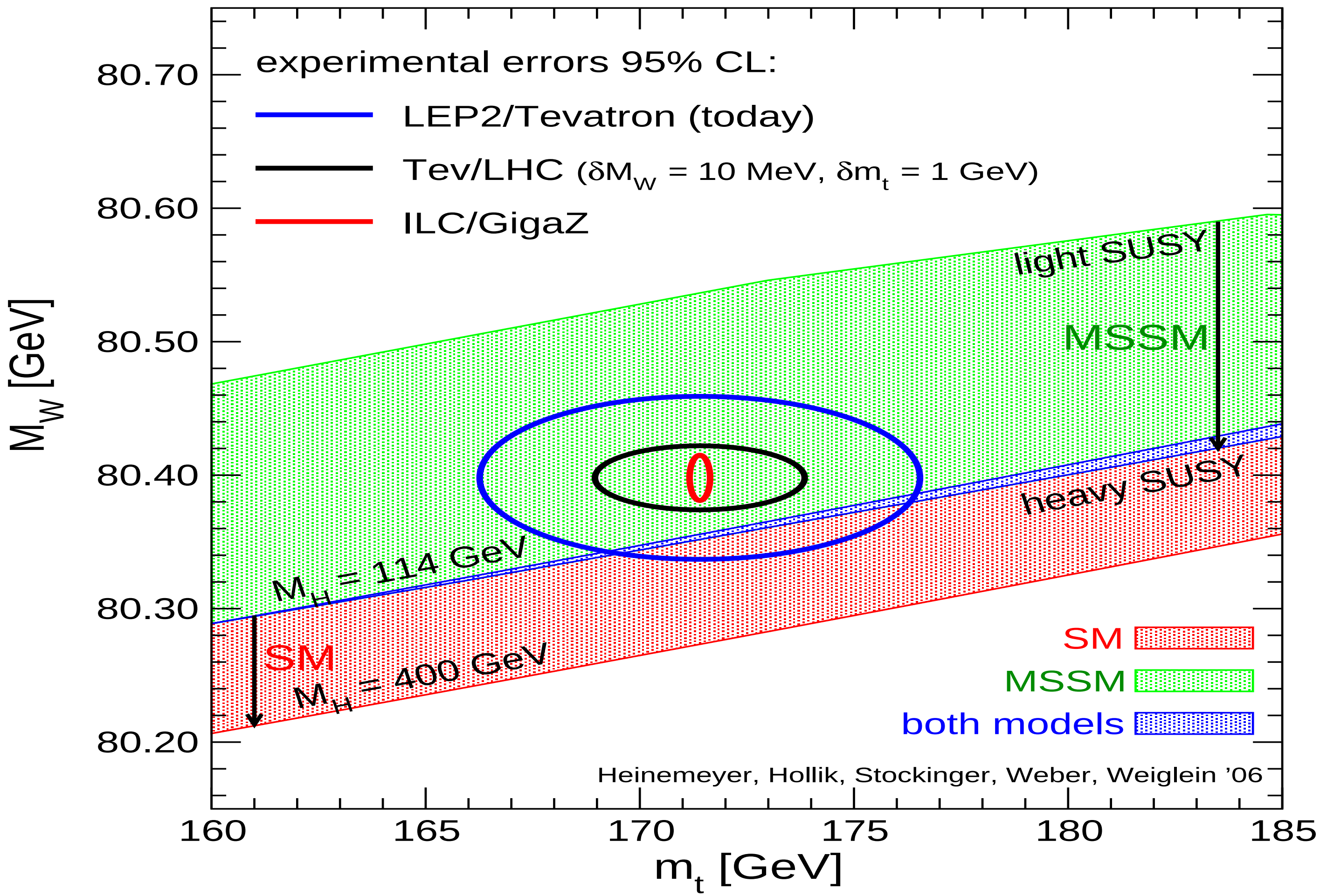
Some issues

- Treatment of large sets of (not necessarily consistent) data ($\Delta\chi^2$)
- Functional forms vs. neural networks. Pumplin [09/09]: relation to $\Delta\chi^2$
- Quantification of also theoretical uncertainties (cf. values of α_s)

Some very low uncertainties given for W/Z and $t\bar{t}$ production at the LHC . . .

Precision physics at the LHC: W-boson mass

M_W as function of τ_μ, M_Z, ... can discriminate between theories



More work, also on partons, required to make the black ellipse happen