# A recursive reduction of tensor Feynman integrals

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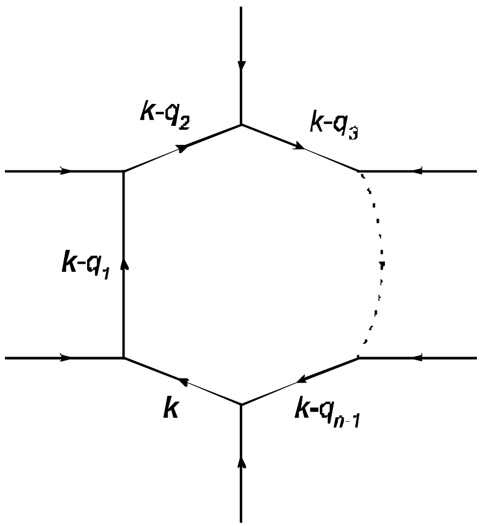
Based on [**arXiv:hep-ph/0907.2115]**

# Motivation and goals

- Recent years have seen the emergence of first results for 2 $\rightarrow$ 4 scattering processes
- One of the challenges posed is the need to compute one-loop tensor integrals with up to 6 legs
- None of available numerical packages is with stable handling in mixed cases (massive and massless particles inside the loop (external lightlike))
- We provide numerical results from compact analytic formulas for the complete reduction of tensor integrals to scalar master integrals
- Implementation to Fortran code

# Tensor reduction (Notations)

We consider one-loop, (N)-point tensor integrals of rank R in d-dimensional space-time,




$$J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1\ldots\nu_N)=\int\frac{d^dk}{i\pi^{d/2}}\frac{k_{\mu_1}\ldots k_{\mu_R}}{D_1^{\nu_1}\ldots D_N^{\nu_N}}$$

with propagator denominators:

$$D_j=(k-q_j)^2-m_j^2+i\varepsilon$$

# Notations continue …

Where:

$$
()_N = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix} = -2^{N-1} \times \begin{vmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_{N-1} \\ q_2 \cdot q_1 & q_2 \cdot q_2 & \cdots & q_2 \cdot q_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N-1} \cdot q_1 & q_{N-1} \cdot q_2 & \cdots & q_{N-1} \cdot q_{N-1} \end{vmatrix}
$$

An $(N+1)\times(N+1)$ matrix known as the modified Cayley determinant (D.B. Melrose, Nuovo Cim. **40** (1965) 181)

with coefficients:

$$
Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \ldots N)
$$

# Notations continue …

$$\begin{pmatrix} i \\ j \end{pmatrix}_N = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & \cdots & Y_{1j} & \cdots & Y_{1n} \\ 1 & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & Y_{i1} & \cdots & Y_{ij} & \cdots & Y_{in} \\ 1 & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & Y_{n1} & \cdots & Y_{nj} & \cdots & Y_{nn} \end{vmatrix}$$

# Hexagons

- The master formula for hexagons is:

$$I_6^{\mu_1\ldots\mu_{R-1}\rho} = I_6^{\mu_1\ldots\mu_{R-1}}\bar{Q}_0^{\rho} - \sum_{s=1}^{6} I_5^{\mu_1\ldots\mu_{R-1},s}\bar{Q}_s^{\rho}$$

- Due to: $\bar{Q}_0^{\mu} = \sum_{i=1}^{6} q_i^{\mu} \frac{\binom{00}{0i}_6}{\binom{0}{0}_6} \equiv 0,$ using $\bar{Q}_s^{\mu} = \sum_{i=1}^{6} q_i^{\mu} \frac{\binom{0s}{0i}_6}{\binom{0}{0}_6}$ , $s = 1\ldots6.$

- We can directly connect hexagons with lower rank pentagons

# Pentagons

- The case of pentagons

$$I_5^{\mu_1\ldots\mu_{R-1}\mu} = I_5^{\mu_1\ldots\mu_{R-1}} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\ldots\mu_{R-1},s} Q_s^{\mu}$$

$$Q_0^{\mu} = \sum_{i=1}^{n} q_i^{\mu} \frac{\begin{pmatrix} 0 \\ i \end{pmatrix}_n}{\left(\ \right)_n} \qquad Q_s^{\mu} = \sum_{i=1}^{n} q_i^{\mu} \frac{\begin{pmatrix} s \\ i \end{pmatrix}_n}{\left(\ \right)_n}, \quad s = 0,\ldots,n$$

- In terms of lower rank pentagons and boxes

Similar reduction steps are found to:

J.Fleisher, F. Jegerlehner, O.Tarasov Nucl. Phys. **B566** (2000) 423

T. Binoth at al. JHEP **0510** (2005) 015

A. Denner and S. Dittmaier, Nucl. Phys. B **734** (2006) 62

# Boxes, Triangles, Bubbles

- Things are getting a little more complicated
- We have extra terms

The most composite cases:

$$I_4^{\mu\nu\lambda\rho} = I_4^{\mu\nu\lambda} Q_0^{\rho} - \sum_{t=1}^{4} I_3^{\mu\nu\lambda,t} Q_t^{\rho} - G^{\mu\rho} T^{\nu\lambda} - G^{\nu\rho} T^{\mu\lambda} - G^{\lambda\rho} T^{\mu\nu}$$

$$G^{\mu\rho} = \frac{1}{2} g^{\mu\rho} - \sum_{i,j=1}^{4} q_i^{\mu} q_i^{\rho} \frac{\binom{i}{j}_4}{\left( \, \right)_4} \qquad T^{\nu\lambda} = -V^{\nu} Q_0^{\lambda} + \sum_{t=1}^{4} W^{t,\nu} Q_t^{\lambda} - G^{\nu\lambda} I_4^{[d+]^2}$$

# Boxes, Triangles, Bubbles

The example of 3rd rank Triangle:

$$I_3^{\mu\nu\lambda,t} = I_3^{\mu\nu,t} Q_0^{t,\lambda} - \sum_{u=1}^{4} I_2^{\mu\nu,tu} Q_u^{t,\lambda} + G^{t,\mu\lambda} W^{t,\nu} + G^{t,\nu\lambda} W^{t,\mu}$$

$$W^{t,\mu} = -I_3^{[d+],t} Q_0^{t,\mu} + \sum_{u=1}^{4} I_2^{[d+],tu} Q_u^{t,\mu} \quad G^{t,\mu\nu} = \frac{1}{2} g^{\mu\nu} - \sum_{i,j=1}^{4} q_i^{\mu} q_j^{\nu} \frac{\begin{pmatrix} it \\ jt \end{pmatrix}_4}{\begin{pmatrix} t \\ t \end{pmatrix}_4}$$

# Boxes, Triangles, Bubbles

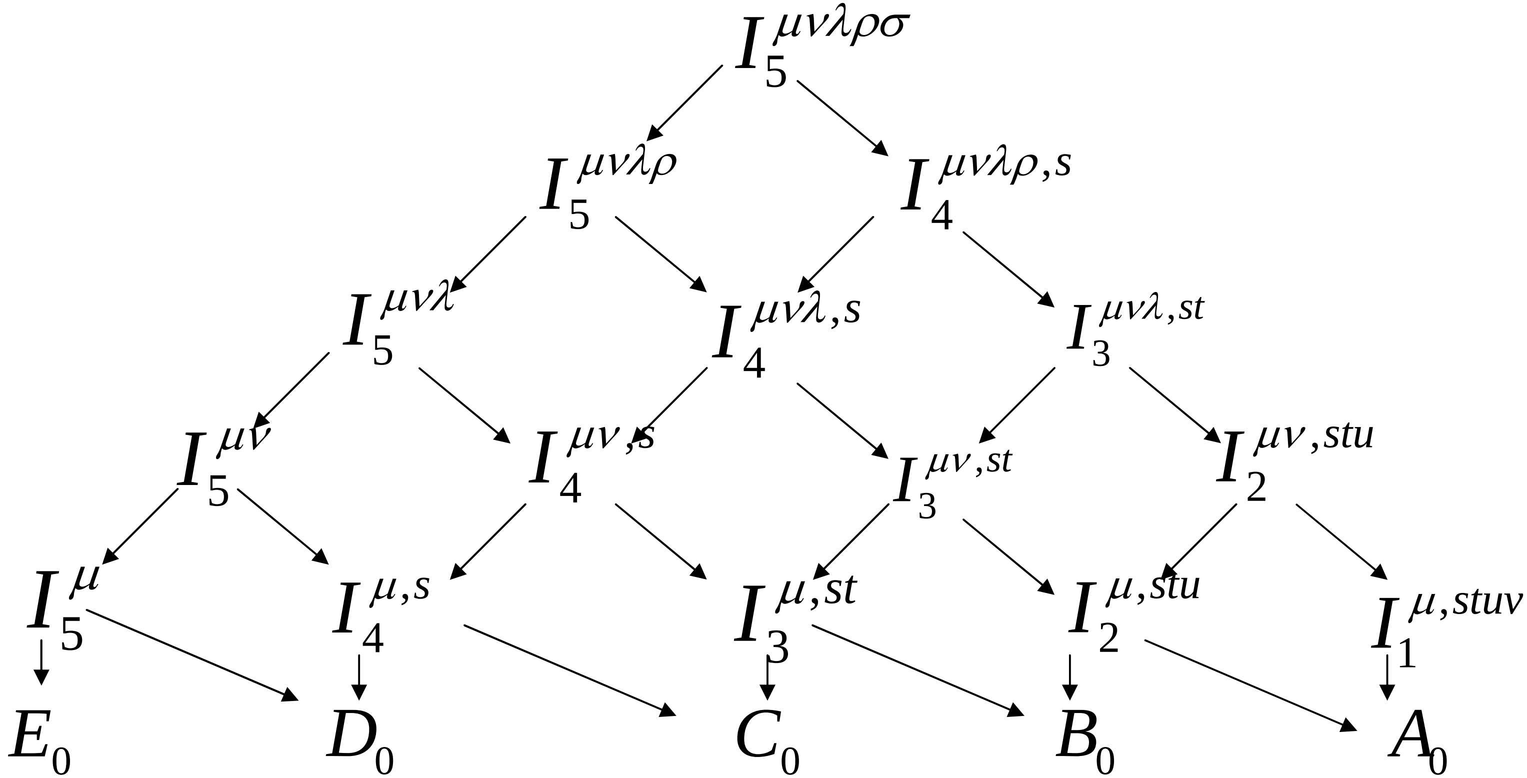
2nd rank bubble:

$$I_2^{\mu\nu,tu} = I_2^{\mu,tu} Q_0^{tu,\nu} - \sum_{v=1}^{4} I_1^{\mu,tuv} Q_v^{tu,\nu} - G^{tu,\mu\nu} I_2^{[d+],tu}$$

$$I_2^{[d+],tu} = \left[ \frac{\begin{pmatrix} 0tu \\ 0tu \end{pmatrix}_4}{\begin{pmatrix} tu \\ tu \end{pmatrix}_4} I_2^{tu} - \sum_{v=1}^{4} \frac{\begin{pmatrix} 0tu \\ vtu \end{pmatrix}_4}{\begin{pmatrix} tu \\ tu \end{pmatrix}_4} I_1^{tuv} \right] \frac{1}{d-1}$$

$$I_1^{\mu,tuv} = -q_i^{\mu} I_{1,i}^{[d+],tuv} = q_i I_1^{tuv}, i \neq t,u,v.$$

# The triangle




Here we have to add some extra terms in the cases of boxes, triangles and bubbles with the exception of 1st rank

# Fortran

- For Tensor integrals, we have a Fortran implementation package (Th. Diakonidis & B. Tausk)

The present implementation includes:

- Six point functions up to rank five (Hexagon.F)
- Five point functions (all 5 ranks) (Pentagon.F)
- Boxes (all 4 ranks) (Box.F)
- Triangles (all 3 ranks) (Triangle.F)
- Bubbles (all 2 ranks) (Bubble.F)

- It is able to output the full result for:

The tensor integrals

- The code so far uses:

QCDLoop (R.K. Ellis and G. Zanderighi)

(Finite part and $1/\varepsilon$ and $1/\varepsilon^2$ terms)

To calculate the scalar master integrals after the reduction

- It can be adapted to any Fortran package for 1,2,3,4 point functions

- A lot of cross checks have been done so far (shown after) and we also cross checked the results with an independent code by Peter Uwer

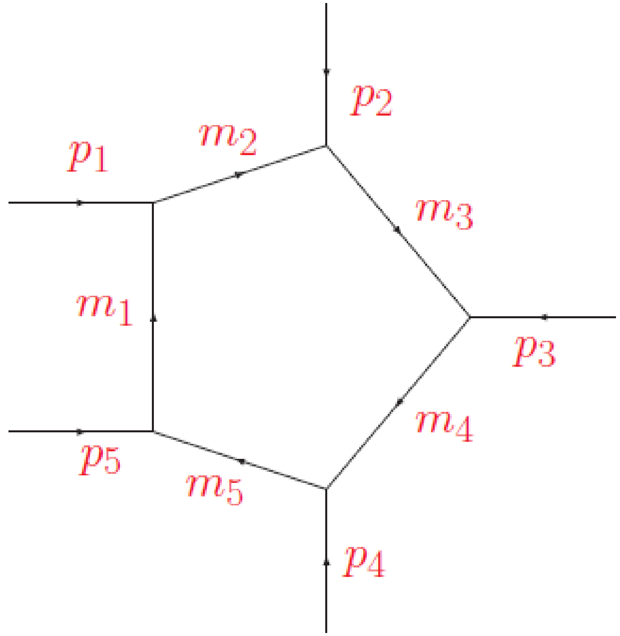
# Starting from a pentagon

For the randomly chosen phase space point:

| | | | | |
|---|---|---|---|---|
| $p_1$ | 5.00000000000 E+00 | 0.0000000000 E+00 | 0.00000000000 E+00 | 4.00000000000 E+00 |
| $p_2$ | 5.00000000000 E+00 | 0.0000000000 E+00 | 0.00000000000 E+00 | -4.00000000000 E+00 |
| $p_3$ | − 0.30770034895 E+01 | 0.5359484673 E+00 | −0.37447035150 E+00 | − 0.20120057390 E+00 |
| $p_4$ | − 0.34048537280 E+01 | 0.2184763540 E-01 | − 0.10479394969 E+01 | 0.12224460727 E+01 |
| $p_5$ | − 0.35181427825 E+01 | − 0.5577961027 E+00 | 0.14224098484 E+01 | − 0.10212454988 E+01 |
| $m_1 = 0.0,\quad m_2 = 2.0,\quad m_3 = 3.0,\quad m_4 = 4.0,\quad m_5 = 5.0$ | | | | |

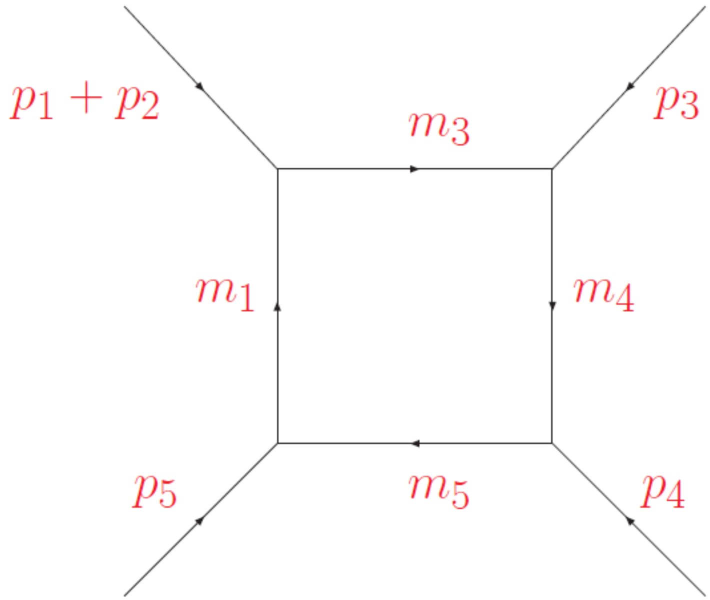
A mixed case of massless and massive particles

# Pentagon



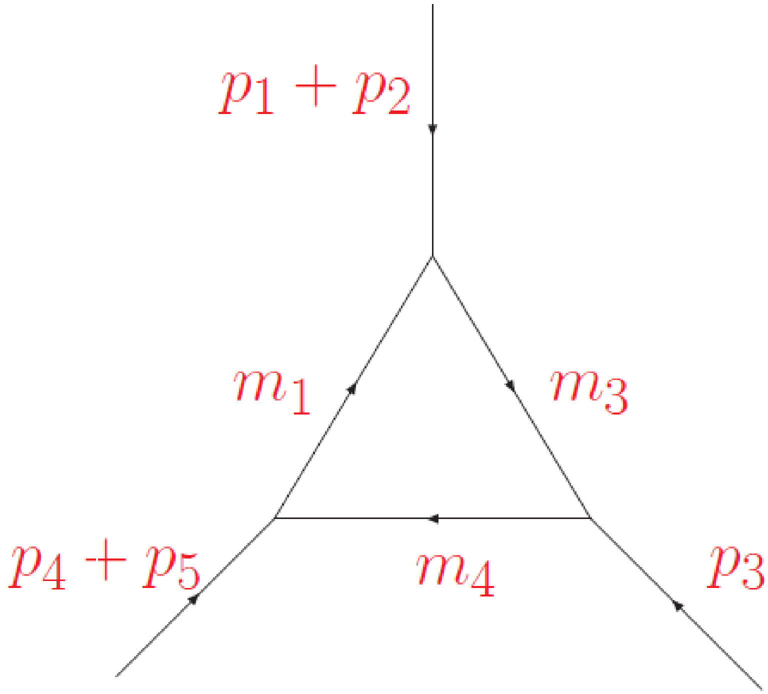

| | *Pentagon.F* |
|---|---|
| $E^2$ | (2.80450709388539E-05 , -1.08461817406464E-05) |
| $E^{12}$ | ( -5.41333978667301E-06 , 6.26985967678899E-06) |
| $E^{232}$ | (-1.20374858970726E-04 , 4.07974751672555E-04) |
| $E^{0321}$ | (-9.11194535703727E-06 , 4.39187998675819E-05) |
| $E^{01230}$ | (4.37928367160152E-05 , -2.18183151665913E-04) |

# Box case



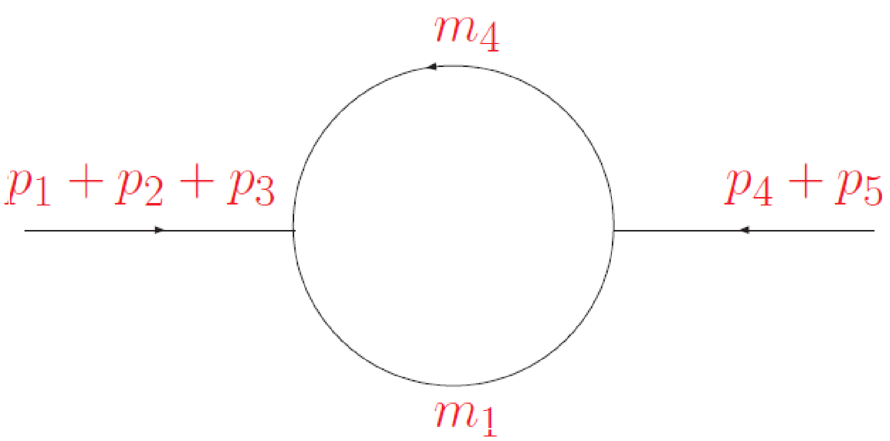

| | $Box.F$ | $LoopTools$ |
|---|---|---|
| $D^1$ | (6.81403420828588E-03 , -5.74298462683219E-03) | (6.81403420828 47463E-03 , -5.74298462683 24187E-03 ) |
| $D^{33}$ | (2.40138809967981E-03 , 1.11591328775015E-02) | (2.40138809968 03092E-03,1.11591328775 00448E-02) |
| $D^{212}$ | ( -1.69702786278243E-03 , -2.83731121595478E-03) | (-1.69702786277 00630E-03,-2.83731121599 62330E-03) |
| $D^{0123}$ | (-1.92190388316994E-04 , -4.04730302413490E-04) | (-1.92190386933 01300E-04,-4.04730301877 72325E-04) |

# Triangle




| | *Triangle.F* | *LoopTools* |
|---|---|---|
| $C^2$ | (2.44757827793318E-04 , -7.50688449850356E-03) | (2.4475782779342707E-04,-7.5068844985030472E-03) |
| $C^{01}$ | (-1.28259813172255E-02 , -6.73809718907549E-02) | (-1.2825981317215014E-02,-6.7380971890795340E-02) |
| $C^{133}$ | (-7.00360822297110E-02 , 7.24628606014397E-02) | (-7.0036082229746830E-02,7.2462860601566081E-02) |

# Bubble




| | *Bubble.F* | *LoopTools* |
|---|---|---|
| $B^3$ | (-0.141525070262337E+00 , 0.138870631815383E+00) | (-0.1415250702623366,0.1388706318153829) |
| $B^{12}$ | (0.102490343329085E+00 , -6.12154531068256E-02) | (0.1024903433290848,-6.1215453106825706E-02) |

# Some sample results for hexagons

For the randomly chosen phase space point given by:

$$
\begin{aligned}
p_1 &= (0.21774554E+03, \quad 0, \quad 0, \quad 0.21774554E+03) \\
p_2 &= (0.21774554E+03, \quad 0, \quad 0, \quad -0.21774554E+03) \\
p_3 &= (-0.20369415E+03, \quad -0.47579512E+02, \quad 0.42126823E+02, \quad 0.84097181E+02) \\
p_4 &= (-0.20907237E+03, \quad 0.55215961E+02, \quad -0.46692034E+02, \quad -0.90010087E+02) \\
p_5 &= (-0.68463308E+01, \quad 0.53063195E+01, \quad 0.29698267E+01, \quad -0.31456871E+01) \\
p_6 &= (-0.15878244E+02, \quad -0.12942769E+02, \quad 0.15953850E+01, \quad 0.90585932E+01) \\
& \quad m_1 = 110.0, \; m_2 = 120.0, \; m_3 = 130.0, \; m_4 = 140.0, \; m_5 = 150.0, \; m_6 = 160.0
\end{aligned}
$$

# Results for scalar, vector and 2nd rank six point functions:

| | | RESULTS | |
|---|---|---|---|
| | | REAL | IM |
| | | $F_0$ | |
| | | -0.223393E-18 | -0.396728E-19 |
| $\mu$ | | $F^\mu$ | |
| 0 | | 0.192487E-17 | 0.972635E-17 |
| 1 | | -0.363320E-17 | -0.11940E-17 |
| 2 | | 0.365514E-17 | 0.106928E-17 |
| 3 | | 0.239793E-16 | 0.341928E-17 |
| $\mu$ | $\nu$ | $F^{\mu\nu}$ | |
| 0 | 0 | 0.599459E-14 | -0.114601E-14 |
| 0 | 1 | 0.323869E-15 | 0.423754E-15 |
| 0 | 2 | -0.294252E-15 | -0.375481E-15 |
| 0 | 3 | -0.255450E-14 | -0.195640E-14 |
| 1 | 1 | -0.164562E-14 | -0.993796E-16 |
| 1 | 2 | 0.920944E-16 | 0.706487E-17 |
| 1 | 3 | 0.347694E-15 | -0.127190E-16 |
| 2 | 2 | -0.163339E-14 | -0.994148E-16 |
| 2 | 3 | -0.341773E-15 | 0.818678E-17 |
| 3 | 3 | -0.413909E-14 | 0.670676E-15 |

# 3rd rank 6 point functions

| $\mu$ | $\nu$ | $\lambda$ | $F^{\mu\nu\lambda}$ |
|---|---|---|---|
| 0 | 0 | 0 | – 0.227754 E–11 – i 0.267244 E–12 |
| 0 | 0 | 1 | 0.140271 E–13 – i 0.119448 E–12 |
| 0 | 0 | 2 | – 0.201270 E–13 + i 0.101968 E–12 |
| 0 | 0 | 3 | 0.102976 E–12 + i 0.624467 E–12 |
| 0 | 1 | 1 | 0.183904 E–12 + i 0.142429 E–12 |
| 0 | 1 | 2 | – 0.131028 E–13 – i 0.610343 E–14 |
| 0 | 1 | 3 | – 0.543316 E–13 – i 0.158809 E–13 |
| 0 | 2 | 2 | 0.181352 E–12 + i 0.141686 E–12 |
| 0 | 2 | 3 | 0.506408 E–13 + i 0.163568 E–13 |
| 0 | 3 | 3 | 0.600542 E–12 + i 0.130733 E–12 |
| 1 | 1 | 1 | – 0.563539 E–13 + i 0.178403 E–13 |
| 1 | 1 | 2 | 0.210641 E–13 – i 0.584990 E–14 |
| 1 | 1 | 3 | 0.120482 E–12 – i 0.574688 E–13 |
| 1 | 2 | 2 | – 0.201182 E–13 + i 0.620591 E–14 |
| 1 | 2 | 3 | – 0.686164 E–14 + i 0.205457 E–14 |
| 1 | 3 | 3 | – 0.447329 E–13 + i 0.193180 E–13 |
| 2 | 2 | 2 | 0.582201 E–13 – i 0.163889 E–13 |
| 2 | 2 | 3 | 0.119659 E–12 – i 0.570084 E–13 |
| 2 | 3 | 3 | 0.457464 E–13 – i 0.181141 E–13 |
| 3 | 3 | 3 | 0.557081 E–12 – i 0.374359 E–12 |

# $4^{th}$ rank 6-point

| | | | | REAL | IM |
|---|---|---|---|---|---|
| $\mu$ | $\nu$ | $\lambda$ | $\rho$ | $F^{\mu\nu\lambda\rho}$ | |
| 0 | 0 | 0 | 0 | 0.666615D-09 | 0.247562D-09 |
| 0 | 0 | 0 | 1 | -0.200049D-10 | 0.294036D-10 |
| 0 | 0 | 0 | 2 | 0.200975D-10 | -0.237333D-10 |
| 0 | 0 | 0 | 3 | 0.645477D-10 | -0.162236D-09 |
| 0 | 0 | 1 | 1 | -0.116956D-10 | -0.516760D-10 |
| 0 | 0 | 1 | 2 | 0.160357D-11 | 0.222284D-11 |
| 0 | 0 | 1 | 3 | 0.792692D-11 | 0.729502D-11 |
| 0 | 0 | 2 | 2 | -0.111838D-10 | -0.513133D-10 |
| 0 | 0 | 2 | 3 | -0.681086D-11 | -0.708933D-11 |
| 0 | 0 | 3 | 3 | -0.804454D-10 | -0.801909D-10 |
| 0 | 1 | 1 | 1 | 0.100498D-10 | -0.151735D-13 |
| 0 | 1 | 1 | 2 | -0.348984D-11 | -0.195436D-12 |
| 0 | 1 | 1 | 3 | -0.211111D-10 | 0.295212D-11 |
| 0 | 1 | 2 | 2 | 0.357455D-11 | 0.662809D-14 |
| 0 | 1 | 2 | 3 | 0.121595D-11 | -0.807388D-13 |
| 0 | 1 | 3 | 3 | 0.825803D-11 | -0.142086D-11 |
| 0 | 2 | 2 | 2 | -0.958961D-11 | -0.585948D-12 |

| | | | | REAL | IM |
|---|---|---|---|---|---|
| $\mu$ | $\nu$ | $\lambda$ | $\rho$ | $F^{\mu\nu\lambda\rho}$ | |
| 0 | 2 | 2 | 3 | -0.209232D-10 | 0.289031D-11 |
| 0 | 2 | 3 | 3 | -0.802359D-11 | 0.994701D-12 |
| 0 | 3 | 3 | 3 | -0.102576D-09 | 0.378476D-10 |
| 1 | 1 | 1 | 1 | -0.246426D-10 | 0.276326D-10 |
| 1 | 1 | 1 | 2 | 0.915670D-12 | -0.660629D-12 |
| 1 | 1 | 1 | 3 | 0.303529D-11 | -0.287480D-11 |
| 1 | 1 | 2 | 2 | -0.822697D-11 | 0.919635D-11 |
| 1 | 1 | 2 | 3 | -0.116294D-11 | 0.100024D-11 |
| 1 | 1 | 3 | 3 | -0.146918D-10 | 0.183799D-10 |
| 1 | 2 | 2 | 2 | 0.908296D-12 | -0.654735D-12 |
| 1 | 2 | 2 | 3 | 0.109510D-11 | -0.100875D-11 |
| 1 | 2 | 3 | 3 | 0.717342D-12 | -0.557293D-12 |
| 1 | 3 | 3 | 3 | 0.450661D-11 | -0.485065D-11 |
| 2 | 2 | 2 | 2 | -0.245154D-10 | 0.274313D-10 |
| 2 | 2 | 2 | 3 | -0.318500D-11 | 0.279750D-11 |
| 2 | 2 | 3 | 3 | -0.146317D-10 | 0.182912D-10 |
| 2 | 3 | 3 | 3 | -0.477335D-11 | 0.477368D-11 |
| 3 | 3 | 3 | 3 | -0.730168D-10 | 0.112865D-09 |

# More results (massless case)

For the phase space point given by:

p1 =(0.5, 0, 0, 0.5)
p2 =(0.5, 0, 0, -0.5)
p3 =(-0.19178191,-0.12741180,-0.08262477,-0.11713105)
p4 =(-0.33662712, 0.06648281, 0.31893785, 0.08471424)
p5 =(-0.21604814, 0.20363139,-0.04415762,-0.05710657)
p6 =-(p1+p2+p3+p4+p5)

M1=0, M2=0, M3=0, M4=0, M5=0, M6=0

Golem95: T.Binoth, J.-Ph.Guillet, G. Heinrich, E.Pilon, T.Reiter [arXiv:hep-ph/0810.0992]

# Comparisons with golem95 (for 5th rank)

| | *Hexagon.F* | *Golem95* |
|---|---|---|
| $F^{03121}$ | ( 0.158428986740235E+00 , 0.416706979843194E-01 ) | (0.158428980552600E+00 , 0.416706995132716E-01 ) |
| $F^{11020}$ | (-0.143913859903552E+01 , -0.164647048275408E+00 ) | (-0.143913852754709E+01 , -0.164647075385477E+00) |
| $F^{20200}$ | (0.242928799509288E+02 , 0.555041844207877E+02 ) | (0.242928775936564E+02 , 0.555041824180155E+02 ) |
| $F^{22130}$ | (0.225563941055782E+00 , 0.231928571404353E+00 ) | (0.225563949300093E+00 , 0.231928509918651E+00 ) |
| $F^{33333}$ | (0.244568134868438E+00 , 0.740146041525474E+00) | (0.244568138432017E+00 , 0.740146095196997E+00) |

A good agreement (8 digits)

(QCDLoop was used for the scalar master integrals)

# Conclusions

- A new analytical recursive reduction method of one-loop tensor integrals down to scalar master integrals has been described
- The connection of $n$-point $R$-rank tensor integrals with $(n,R-1)$ and $(n-1,R-1)$ integrals was made evident
- Reduction formulas have been implemented in a Fortran program
- Results and cross checks with other programs were shown